Probability Spaces

Motivation

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2. Builds a necessary foundation for statistics
    - Allows you to make inferences about populations or processes from limited data.

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1. Sample Spaces
2. Probability
3. Combinatorial Probability

Some Definitions

Definition

An **Experiment** in an activity whose outcome cannot be *a priori* determined with certainty.

Definition

A **sample space**, often denoted S is the set (collection) of all possible outcomes in an experiment. A sample space is called **discrete** if the number of possible outcomes is countable (or finite), and **continuous** if the number of possible outcomes is uncountable (e.g. an interval)

Definition

An **event** is a collection of outcomes (a subset of S)

Example: Coin Toss

Example: Rolling Dice

Example: Drawing Cards

Union and Intersection

Let A and B be events. Then:

- The collection of outcomes in A **or** B is called the **union** of A and B, denoted $A \cup B$.
- The collection of all elements that are in A **and** B is called the **intersection** of A and B, denoted $A \cap B$.

Examples

Subsets and Complements

- If each outcome in a set A is also in B, we say that A is a **subset** of B, denoted $A \subset B$.
- The **empty set** $\emptyset$ is the set which contains no elements.
  - By convention, $\emptyset$ is considered a subset of all other events.
- The **complement** af an event A is the set of all outcomes that are not in A and is denoted A^c or A'.

Examples

Mutually Exclusive Events

- Two events A and B are called **mutually exclusive** (or disjoint) if they share no outcomes
  - A and B are mutually exclusive if and only if $A \cap B = \emptyset$

Examples

Partitioning the Sample Space

Some Properties of Set Operations

- Commutative and associative laws:

- Distributive laws:

- DeMorgan's laws:

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What is Probability?

A **probability measure** assigns probabilities to events. The probability of an event A is denoted

$$P(A)$$

Example: Schrödinger's Cat

Interpretation

One way to think about it ("Frequency Interpretation"):

The long-run proportion of the time an event will happen under repeated experiments:

$$P(A) = \lim_{t \to \infty} \frac{\text{number of times } A \text{ occurs}}{t}$$

Interpretation

Another way to think about it ("Subjective Interpretation"):

> The "relative likelihood" of an event from an observer's perspective.

Properties of Probabilities

We can assign probabilities to events based on the following rules:

1. $P(A) \geq 0$ for any event A.
2. $P(S) = 1$.
3. If $(A_1, A_2, A_3, \ldots)$ is a sequence of mutually exclusive events, then $P(\bigcup_i A_i) = P(A_1 \cup A_2 \cup A_3, \cup \ldots) = \sum_i P(A_i) = P(A_1) + P(A_2) + P(A_3) + \cdots$.

Complements and the Empty Set

Monotonicity

The Addition Rule

Example

A party organizer is planning an event that will be outdoors as long as it is not windy or rainy. If it is windy or rainy, the event will need to be moved indoors. The weather forecast suggests that the probability that there is a 27% chance of rain, a 36% chance of windy weather, and a 21% chance that it will be both windy and rainy. What is the probability that they will be able to hold the event outdoors?

Example

The probability that a randomly chosen student is a sophomore is .28. The probability that a randomly chosen student is a chemical engineering major is .14. Overall, 36% of students are sophomores or chemical engineering majors. What is the probability that a randomly chosen student is a sophomore majoring in chemical engineering?

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Example: Rolling Dice

Finite Probability Spaces

- The simplest probability measure we can construct on a finite probability space assigns equal probability to each outcome. This is sometimes called the (discrete) **uniform distribution**.

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- Finding probabilities of events in this context is referred to as **combinatorial probability**; we find probabilities of events by counting the number of outcomes in the sample space and in events of interest.

Example: Drawing Cards

Counting Rules

The *mn* **rule** says that if $A = \{a_1, \ldots, a_m\}$ is a set with m elements and $B = \{b_1, \ldots, b_n\}$ is a set with n elements, then there are mn ways to choose one element from A and one element from B:

Note: The *mn* rule generalizes to sampling from any number of sets

Example: Getting Dressed

Example: Coin Toss

Counting Rules

$n!$ counts the number of ways to order n elements:

Counting Rules

A **permutation**, denoted ${}_nP_k$, counts the number of ways to choose k elements from an n element set, where order matters:

Example: Olympic Swimming

Counting Rules

A **combination**, denoted ${}_nC_k$ or $\binom{n}{k}$, counts the number of ways to choose k elements from an n element set, where order **does not** matter:

Example: Poker Hands