Asymptotic Analysis (Ch. 3 from Cormen)

When we talk about running time, we will use asymptotic analysis. **The following definitions are crucial. Burn them into your memory**:

Definition 0.1. *Let $\mathcal{F}$ be the set of functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let $g \in \mathcal{F}$. Then*

1. $O(g(n)) = \{f \in \mathcal{F} : \exists c, n_0 > 0, \forall n \geq n_o, f(n) \leq cg(n)\}$
2. $\Omega(g(n)) = \{f \in \mathcal{F} : \exists c, n_0 > 0, \forall n \geq n_o, f(n) \geq cg(n)\}$
3. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

We use these in the following way:
For functions $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we write

- $f(n) = O(g(n))$ to mean that f grows no faster than g, i.e. the growth of g is an upper bound to the growth of f.
- $f(n) = \Omega(g(n))$ to mean that f grows at least as fast as g, i.e. g is a lower bound to the growth of f.
- $f(n) = \Theta(g(n))$ to mean that f grows as fast as g.

Here are two more definition, also worth memorizing:

Definition 0.2. *Let $\mathcal{F}$ be the set of functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let $g \in \mathcal{F}$. Then*

1. $o(g(n)) = O(g(n)) \setminus \Theta(g(n))$
2. $\omega(g(n)) = \Omega(g(n)) \setminus \Theta(g(n))$

We use these in the following way:
For functions $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we write

- $f(n) = o(g(n))$ to mean that f grows noticeably slower than g.
- $f(n) = \omega(g(n))$ to mean that f grows noticeably faster than g.

The following theorem is useful for when we have a good understanding of how a function grows:

Theorem 0.3. *Let* $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ *be monotonically increasing, i.e for all* $a, b \in \mathbb{R}^+$ *with* $a < b$ *we have that*

$$f(a) \leq f(b)$$

and

$$g(a) \leq g(b).$$

Then

1. *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ *then* $f(n) = o(g(n))$.
2. *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$ *then* $f(n) = \Theta(g(n))$.
3. *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ *then* $f(n) = \omega(g(n))$.

Proof. By authority. □

To use this theorem, you have to be able to solve limits. Remember L'Hopitals rule from calculus:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

We also have the following helpful lemma:

Lemma 0.4. *Let* $f, g : \mathbb{R}^+ \to \mathbb{R}_{>1}$ *be monotonically increasing.*
If $f(n) = O(g(n))$ *then* $\ln(f(n)) = O(\ln(g(n)))$.

Proof. Let $f, g : \mathbb{R}^+ \to \mathbb{R}_{>1}$ be monotonically increasing functions such that $f(n) = O(g(n))$.
So there exists $c, n_0 > 0$ such that $\forall n > n_0, f(n) \leq cg(n)$.
$\ln(x)$ is monotonically increasing, so for all $0 < a \leq b$ we have that

$$\ln(a) \leq \ln(b).$$

Therefore, for all $n \geq n_0$, we have that

$$\ln(f(n)) \leq \ln(cg(n)) \qquad \text{for all } n \geq n_0$$

Case 1: $c \leq 1$.
Then $\ln(c) \leq 0$.
So

$$\begin{aligned}
\ln(f(n)) &\leq \ln(cg(n)) && \text{for all } n \geq n_0 \\
&= \ln(c) + \ln(g(n)) \\
&\leq \ln(g(n)) && \text{for all } n \geq n_0
\end{aligned}$$

Case 2: $c > 1$.
Then $\ln(c) > 0$.
Case 2.1: There exists $m_0 > 0$ such that $\forall n \geq m_0, \ln(g(n)) \geq 1$.

Then we have

$$\begin{aligned}
\ln(f(n)) &\leq \ln(c) + \ln(g(n)) && \text{for all } n \geq n_0 \\
&\leq \ln(c)\ln(g(n)) + \ln(g(n)) && \text{for all } n \geq \max\{n_0, m_0\} \\
&= (\ln(c) + 1)\ln(g(n)) && \text{for all } n \geq \max\{n_0, m_0\}
\end{aligned}$$

Case 2.2: $\forall n > 0, \ln(g(n)) < 1$.
Since $g(n) : \mathbb{R}^+ \to \mathbb{R}_{>1}$ is monotonically increasing we have that

$$1 < g(1) \leq g(n) \text{ for all } n > 1$$

therefore

$$0 < \ln(g(1)) \leq \ln(g(n)) \text{ for all } n > 1.$$

there

$$1 \leq \frac{\ln(g(n))}{\ln(g(1))} \text{ for all } n > 1.$$

Therefore,

$$\begin{aligned}
\ln(f(n)) &\leq \ln(c) + \ln(g(n)) && \text{for all } n \geq n_0 \\
&\leq \ln(c)\frac{\ln(g(n))}{\ln(g(1))} + \ln(g(n)) && \text{for all } n \geq n_0 \\
&= \left(\frac{\ln(c)}{\ln(g(1))} + 1\right)\ln(g(n)) && \text{for all } n \geq n_0
\end{aligned}$$

Conclusion: Letting

$$c' = \max\left\{1, 1 + \ln(c), \frac{\ln(c)}{\ln(g(1))} + 1\right\}$$

and

$$n_0' = \max\{n_0, m_0\}$$

we have that $\ln(f(n)) \leq c' \ln(g(n))$ for all $n > n_0'$.
Therefore, $\ln(f(n)) = O(\ln(g(n)))$. □

This lemma will be useful to us because it gives a necessary condition for $f(n) = O(g(n))$ that we can take advantage of in proof by contradiction.