Adam Gluck CSE 2321 Exam 1

Prepositional Logic

- Symbols
  - $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Tautology: Always True
- Contradiction: Always False
- Contingency: Not tautology or contradiction
- English Prepositions
  - I wear a hat if it's sunny
    - $sunny \Rightarrow hat$
  - I wear a hat only if it's sunny
    - $hat \Rightarrow sunny$
- P and Q are logically equivalent if $P \Leftrightarrow Q$ is a tautology
- Double Negation
  - $(\neg(\neg P)) \equiv P$
- Commutative Laws
  - $(P \vee Q) \equiv (Q \vee P)$
  - $(P \wedge Q) \equiv (Q \wedge P)$
- Associative Laws
  - $((P \wedge Q) \vee R) \equiv (P \vee (Q \vee R))$
- Distributive Laws
  - $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$
  - $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$
- DeMorgans Law
  - $(\neg(P \vee Q)) \equiv ((\neg P) \wedge (\neg Q))$
  - $(\neg(P \wedge Q)) \equiv ((\neg P) \vee (\neg Q))$
- The conditional and the contrapositive
  - $(P \Rightarrow Q) \equiv ((\neg Q) \Rightarrow (\neg P))$
- The inverse and the converse
  - $((\neg P) \Rightarrow (\neg Q)) \equiv (Q \Rightarrow P)$
- The conditional and a not/or expression
  - $(P \Rightarrow Q) \equiv ((\neg P) \vee Q)$

Predicate Logic

- Notation
  - $\forall R(x), P(x) \equiv \forall x(R(x) \Rightarrow P(x)$
  - $\exists R(x), P(x) \equiv \exists x(R(x) \wedge P(x)$
- Interactions with $\neg$
  - $\neg\forall x \in S, P(x) \equiv \exists x \in S, \neg P(x)$
  - $\neg\exists x \in S, P(x) \equiv \forall x \in S, \neg P(x)$
- There is at most one $P(X)$ in S
  - $\exists x \in S, \forall y \in S, \left(\neg(x \neq y) \Rightarrow (P(x) \wedge \neg P(y))\right)$
- There is exactly one $P(X)$ in S
  - $\exists x \in S, \left(P(x) \wedge (\forall y \in S, P(y) \Rightarrow (x = y))\right)$
- Is a prime number
  - $P(x) = \forall y, z \in \mathbb{N}, (x = yz) \Rightarrow \neg(y = 1 \Leftrightarrow z = 1)$

Sets

- Notation
  - $S_1 = \{x: P(x)\}$
  - $S_2 = \{x \in S: P(x)\}$
- Schema of Separation
  - If ϕ is a predicate, then for any set X there exists a set $Y = \{x \in X: \phi(x)\}$
- X equals Y
  - $\forall u(u \in X \Leftrightarrow u \in Y)$
- $\emptyset$ is a set with no elements
- $|X|$ is cardinality of X
  - Number of elements
- X is a subset of Y
  - $X \subseteq Y \equiv \forall x(x \in X \Rightarrow x \in Y)$
- X is a proper subset of Y
  - $(X \subseteq Y) \wedge (X \neq Y)$
- Union of X and Y
  - $Z = X \cup Y \equiv \forall u(u \in Z \Leftrightarrow u \in X \vee U \in Y)$
- Intersection of X and Y
  - $X \cap Y \equiv \{u \in X: u \in Y\}$
- Difference of X and Y
  - $X \backslash Y \equiv \{u \in X: u \notin Y\}$
- Cartesian Product
  - $\{(u, v): u \in X \wedge v \in Y\}$
- Power Set
  - $Pow(X) \equiv \{u: u \subseteq X\}$
  - $|Pow(x)| = 2^{|x|}$

Graphs

- A directed graph is an order pair (V, E) of sets
  - $E \subseteq \{(v, w): v, w \in V \wedge v \neq w\}$
- A undirected graph is an order pair (V, E) of sets
  - $E \subseteq \{\{v, w\}: v, w \in V \wedge v \neq w\}$
- Maximum number of edges in an undirected graph with *n* vertices
  - $\frac{(|v|)(|v|-1)}{2}$
- Maximum number of edges in an directed graph with *n* vertices
  - $(|v|)(|v| - 1)$
- Degree of a vertex (numbers of edges connected)
  - undirected graph
    - $deg(v) = |\{w: \{v, w\} \in E\}|$
  - directed graph
    - $indeg(v) = |(w: (w, v) \in E\}|$
    - $outdeg(v) = |(w: (v, w) \in E\}|$
    - $deg(v) = indeg(v) + outdeg(v)$

- Path in a graph
  - Directed
    - $(v_i, v_{i+1}) \in E$
  - Undirected
    - $\{v_i, v_{i+1}\} \in E$
  - The length of a path is the number of edges
    - if no vertices are repeated it is a simple path
- A cycle is a path where its length is its number of vertices, and $v_0 = v_k$
  - if the path defined by the cycle is a simple path, it is a simple cycle
- Hamiltonian
  - A path is Hamiltonian if it includes every vertex exactly once
  - A cycle is Hamiltonian if it includes every vertex exactly once
- Eulerian
  - A path is Eulerian if it includes every edge exactly once
  - A cycle is Eulerian if it includes every edge exactly once
- An undirected graph $G = (V, E)$ is connected if for all $x, y \in V$ there is a path in G from x to y
- Let $G = (V, E)$ be a connected undirected graph. If every vertex in V has even degree then G has an Eulerian cycle.

Proofs

- Two Essential Features
  - It must be finitely long so you can give it to someone else
  - It must be possible for someone else to check the proof for correctness without needing brilliant flashes of insight
- Direct Proof
  - State the statement you want to prove.
  - Proof.
    - Each line should be a statement or declaration.
    - Each statement should be true.
    - The truth of each statement should be clear from a definition, axiom, lemma, theorem, the previous line, or some combination.
    - Provide a short and precise explanation/justification when needed.
    - Conclusion should mirror the statement we are proving.
- Proof By Induction
  - State the statement you want to prove, typically a statement like ∀n ∈ N, f(n).
  - Proof by Induction Components
    - Base Case:
      - Compute f(n) directly for the base case, the smallest value of n, typically n = 0 or n = 1.
    - Induction Step:
      - Assume the statement is true for an arbitrary value n.
      - Prove the statement is true for n + 1. In other words, we prove here that f(n) ⇒ f(n + 1).
      - Conclusion: State what you have proven
- Proof By Contradiction
  - Works by using the observation that $A \vee \neg A$ is a tautology; if I can prove ¬A is false then I have proven A is true.
  - A proof by contradiction will assume the negation of the statement we actually want to prove, adding it to our set of axioms then use that to prove something obviously false (a contradiction, like 1 = 2). Since we (generally) believe that the axioms and inference rules of mathematics are correct, the contradiction must be coming from the axiom we added
  - Proof By Contradiction Components
    - Assume ¬(statement you want to prove)
    - Find something “obviously false” that would be true as a consequence of that assumption.
    - State the negation of your assumption.