1 Graphs (Appx. B.4 from Cormen)

We have two kinds of graphs that we want to discuss, **directed graphs** and **undirected graphs**.

Definition 1.1. *A directed graph is an order pair* (V, E) *of sets such that*

$$E \subseteq \{(v, w) : v, w \in V \wedge v \neq w\}.$$

An undirected graph is an ordered pair (V, E) *of sets such that*

$$E \subseteq \{\{v, w\} : v, w \in V \wedge v \neq w\}.$$

In either case,

- *elements of* V *are called vertices,*
- *elements of* E *are called edges.*

Question:
What is the maximum number of edges we can have in an undirected graph with n vertices? I.e. let $|V| = n$, what is the maximum value $|E|$ can have?

Question:
What about a directed graph with n vertices?

Definition 1.2. *The degree of a vertex:*
For an undirected graph $G = (V, E)$ *for all* $v \in V$*,*

$$deg(v) = |\{w : \{v, w\} \in E\}|$$

For a directed graph $G = (V, E)$ *for all* $v \in V$*,*

$$\begin{aligned} indeg(v) &= |\{w : (w, v) \in E\}| \\ outdeg(v) &= |\{w : (v, w) \in E\}| \\ deg(v) &= indeg(v) + outdeg(v) \end{aligned}$$

Definition 1.3. *A path in a graph* $G = (V, E)$ *is a sequence* $v_1, v_2, \ldots, v_k \in V$ *such that for all* $i \in \{1, 2, \ldots, k-1\}$*,*

$$(v_i, v_{i+1}) \in E$$

if G *is directed, and*

$$\{v_i, v_{i+1}\} \in E$$

if G *is undirected. The length of the path* $v_1, v_2, \ldots, v_k$ *is* $k-1$*, the number of edges.*
If no vertices are repeated, we say it is a simple path.

Definition 1.4. *A cycle in a graph* $G = (V, E)$ *is a path* $v_0, v_1, v_2, \ldots, v_k$ *such that* $v_0 = v_k$*, and the length of this cycle is* k*.*
If $v_1, v_2, v_3, \ldots, v_k$ *is a simple path, we say* $v_0, v_1, v_2, \ldots, v_k$ *is a simple cycle.*

Definition 1.5. *A path is Hamiltonian it includes every vertex exactly once. A cycle is Hamiltonian if it includes every vertex except the starting vertex exactly once.*

Definition 1.6. *A path or cycle is Eulerian if it includes every edge exactly once.*

Definition 1.7. *An undirected graph* $G = (V, E)$ *is connected if for all* $x, y \in V$ *there is a path in* G *from* x *to* y*.*

Theorem 1.8. *Let* $G = (V, E)$ *be a connected undirected graph.*
If every vertex in V *has even degree then* G *has an Eulerian cycle.*

Proof. What is a proof? □