1 Sets (Appx. B.1, B.2, B.3 from Cormen)

From the lectures on predicates, we have this informal definition of sets:

Definition 1.1. *A **set** is a collection of distinct objects, called **elements**. To indicate that x is an element of set S, we write $x \in S$.*

Our goal in these lectures will be to more precisely define sets and define some useful operations we can perform on sets.

Creating sets

The conventional notation for a set is to list the elements inside braces, like this:

$$\{1, 2, 3\}$$

which give us the set containing 1, 2, and 3.

For larger or infinite sets, we can define them by listing enough elements to establish a pattern, such as

$$\{0, 1, 2, \ldots, 10\}$$

being the set of integers from 0 to 10 and

$$\{0, 1, 2, \ldots\}$$

being the set of natural numbers.
For more complicated sets, the "pattern recognition" kind of set definition is not sufficient.

$$\{0, 1, 2, 3, 5, 8, \ldots\}$$

In these cases, we use *set builder notation*. For this, we use one of the following formats:

$$S_1 = \{x : P(x)\}$$

or

$$S_2 = \{x \in S : P(x)\}$$

where S is some set and $P(x)$ is a predicate.

We can think of this as a sort of iterative structure:

the part before the colon is a variable declaration, and the predicate $P(x)$ is a test that the variable must pass to be included in the set being defined.

$$\{x \in \mathbb{N} : x > 3\}$$

So in the case of S_1, we think of x as a variable that takes on all possible values as we iterate, and those values for which $P(x)$ is true are included in the set S_1.

In S_2, the variable x is restricted to just elements of the set S, and we test each element with the predicate $P(x)$ to see which are included in the set S_2 being defined.

Axiomatic set theory

Although a recent development, since their conception in the late 1800s set have become nearly universally used as the foundational grounding of all mathematics.

This has motivated significant research into making our definition of sets paradox free while maintaining their expressive power.
With the naive definition of sets we give above, the following rule for the definition of sets might seem reasonable (but is it a bad rule)

Axiom 1.2. ***Schema of Comprehension (false).*** *If ϕ is a predicate, then there exists a set $Y = \{x : \phi(x)\}$.*

To see the problem here, consider this:

Observation 1.3. *Consider the set S whose elements are all those (and only those) sets that are not members of themselves, i.e.*

$$S = \{X : X \notin X\}$$

Question: Is S a member of S?

If $S \in S$ then S is not a member of S and therefore $S \notin S$.

If $S \notin S$ then S is a member of S and therefore $S \in S$.

In either case, we have a contradiction.

This is known as **Russell's Paradox**, and leads us to the conclusion that $\{X : X \notin X\}$ is not a set. So we need to be more careful about how we define what is and is not a set.
To fix this paradox, instead of using the Axiom Schema of Comprehension mathematicians use this:

Axiom 1.4. ***Schema of Separation.*** *If ϕ is a predicate, then for any set X there exists a set $Y = \{x \in X : \phi(x)\}$.*

The key difference here is that the ASoC allows picking elements from some paradoxical set of all sets. The ASoS only allows building sets from other known sets.

Axioms and operations on sets

We will not go into the full formal axiomization of sets. Instead, we will discuss some useful axioms and useful operations on sets that produce new sets.

Axiom 1.5. *Let X and Y be sets. We say $X = Y$ are equal when they have all the same elements, i.e. when*

$$\forall u(u \in X \iff u \in Y)$$

Definition 1.6. *For any set X, we say $|X|$ to denote the* cardinality *of X, i.e. the number of elements in X.*

Definition 1.7. *We use* $\emptyset$ *to denote the unique set containing no elements. We call* $\emptyset$ *the* empty set*.*

Definition 1.8. *For any sets* X, Y *we say* $X \subseteq Y$ *(*X *is a subset of* Y*) when*

$$\forall x(x \in X \Rightarrow x \in Y)$$

Definition 1.9. *For any sets* X, Y *we say* $X \subset Y$ *(*X *is a proper subset of* Y*) when*

$$(X \subseteq Y) \wedge (X \neq Y)$$

Please note: Some sources may use $\subset$ to mean $\subseteq$! It is becoming uncommon in modern sources, but older texts and papers do this often. It is important to pay attention to the definitions given in the source you are working with.

Axiom 1.10. *For any sets* X, Y*, there exists a set* $Z = X \cup Y$*, the union of* X *and* Y*, and*

$$\forall u(u \in Z \iff u \in X \vee u \in Y)$$

Definition 1.11. *For any sets* X, Y*, there exists a set* $Z = X \cap Y$*, the intersection of* X *and* Y*, and*

$$Z = \{u \in X : u \in Y\}$$

Why is this a definition and not an axiom?
We can get it from the Axiom of Scheme Separation from before, no need to introduce extra axioms.

Definition 1.12. *For any sets* X, Y*, there exists a set* $Z = X \setminus Y$*, the difference of* X *and* Y*, and*

$$Z = \{u \in X : u \notin Y\}$$

Definition 1.13. *For any sets* X, Y*, there exists a set* $Z = X \times Y$*, the cartesian product of* X *and* Y*, and*

$$Z = \{(u, v) : u \in X \wedge v \in Y\}$$

where (u, v) *is an ordered pair.*

Exercise: Suppose I have sets X, Y and $|X| = n$, $|Y| = m$. Then we can say $|X \times Y| = n * m$?

Axiom 1.14. *For any set X there exists a set $Z = Pow(X)$, the power set of X, and*

$$Z = \{u : u \subseteq X\}$$

Exercise: Suppose I have set X and $|X| = n$. Then what can you say about $|Pow(X)|$?

In the most popular axiomization of set theory (called ZFC) there are 5 more axioms. We do not need these axioms unless we intend to do an intensive study of the foundation of mathematics.