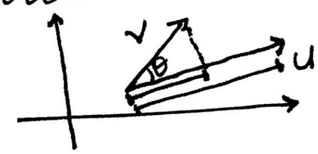
12.3 Dot Products

Def Given two nonzero vectors u and v in two or three dimensions, their dot product is

$$u \cdot v = |u||v| \cos\theta$$

where θ is the angle between u and v with $0 \le \theta \le \pi$.



Observations: if $\theta > \pi/2$ then $u \cdot v < 0$

if $\theta < \pi/2$ then $u \cdot v > 0$

if $\theta = \pi/2$ then $u \cdot v = 0$

Def Two vectors u and v are orthogonal if and only if $u \cdot v = 0$.

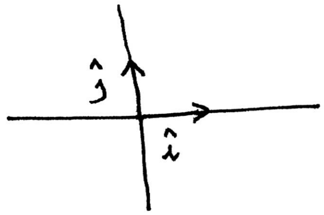
In two or three dimensions, two nonzero orthogonal vectors are perpendicular to each other.

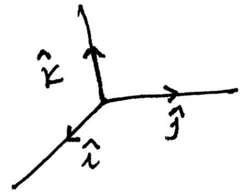
→ $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

Theorem Given two vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$,

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Pf





$\hat{i}\cdot\hat{i} = 1$ $\quad$ $\hat{i}\cdot\hat{j} = 0$ $\quad$ $\hat{j}\cdot\hat{i} = 0$

$\hat{j}\cdot\hat{j} = 1$ $\quad$ $\hat{i}\cdot\hat{k} = 0$ $\quad$ $\hat{k}\cdot\hat{i} = 0$

$\hat{k}\cdot\hat{k} = 1$ $\quad$ $\hat{j}\cdot\hat{k} = 0$ $\quad$ $\hat{k}\cdot\hat{j} = 0$

Observation $\quad u\cdot v = |u|\,|v| \cos\theta = |v|\,|u| \cos\theta$

$\therefore \quad u\cdot v = v\cdot u$

That is, the dot product is commutative.

$u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

$v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\begin{aligned} u\cdot v &= (u_1\hat{i} + u_2\hat{j} + u_3\hat{k})\cdot(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \\ &= u_1v_1(\hat{i}\cdot\hat{i}) + u_1v_2(\hat{i}\cdot\hat{j}) + \dots + (u_2v_2)(\hat{j}\cdot\hat{j}) \\ &\quad + u_2v_3(\hat{j}\cdot\hat{k}) + \dots + u_3v_3(\hat{k}\cdot\hat{k}) \\ &= u_1v_1 + u_2v_2 + u_3v_3 \end{aligned}$$

- $u\cdot v = |u|\,|v| \cos\theta$

  This original definition of dot product is the one we will use when computing <u>angles between vectors</u>:

$$\boxed{\cos\theta = \frac{u\cdot v}{|u|\,|v|}}$$

Example Let $u = \hat{\imath} - 4\hat{\jmath} - 6\hat{k}$ and $v = 2\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

(a) Find the dot product of u and v

$$u \cdot v = (1)(2) + (-4)(-4) + (-6)(2)$$

$$= 2 + 16 - 12 = \underline{\underline{6}}$$

Since $u \cdot v = 6$ is positive we expect the angle between u and v to be less than $\pi/2$

(b) Find the angle between the vectors

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

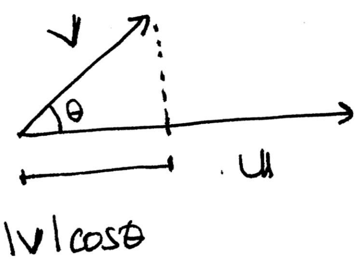
$$|u| = \sqrt{1 + 16 + 36} = \sqrt{53}$$

$$|v| = \sqrt{4 + 16 + 4} = \sqrt{24}$$

$$\therefore \quad \cos\theta = \frac{6}{\sqrt{53}\sqrt{24}} = 0.16823$$

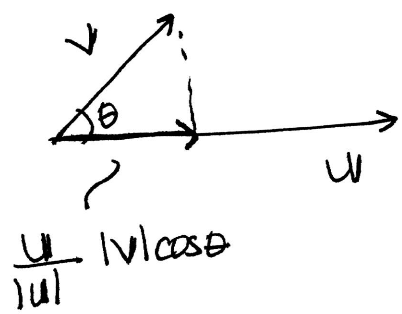
$$\therefore \quad \theta = 1.4 \text{ rad}$$

Orthogonal projections.



to construct a vector that goes along u with magnitude |v| cos θ we do:

$$\frac{u}{|u|}\,|v|\cos\theta$$



Def The <u>orthogonal projection</u> of v onto u, denoted $\text{proj}_u v$, where $v \neq \vec{0}$, is

$$\text{proj}_u v = |v|\cos\theta\left(\frac{u}{|u|}\right)$$

This orthogonal projection can also be computed as follows:

$$\text{proj}_u v = |v|\cos\theta\,\frac{u}{|u|} = \underbrace{|u|\,|v|\cos\theta}_{u\cdot v}\,\frac{u}{|u|^2}$$

$$\therefore \quad \text{proj}_u v = \frac{u\cdot v}{|u|^2}\,u$$

Def The scalar component of $\mathbf{v}$ in the direction of $\mathbf{u}$ is:

$$\text{scal}_{\mathbf{u}}\mathbf{v} = |\mathbf{v}| \cos\theta$$

This scalar component can also be computed as follows:

$$\text{scal}_{u} v = |v| \cos\theta = \frac{|u|\,|v| \cos\theta}{|u|} = \frac{u \cdot v}{|u|}$$

Example Calculate $\text{proj}_u v$ and $\text{scal}_u v$ if

$u = \langle -1, 4 \rangle$

$v = \langle -4, 2 \rangle$

$$\text{proj}_u v = \frac{u \cdot v}{|u|^2}\, u$$

$u \cdot v = (-1)(-4) + (4)(2) = 12$, $|u| = \sqrt{17}$, $|u|^2 = 17$

$$\therefore \quad \text{proj}_u v = \frac{12}{17} \langle -1, 4 \rangle = \left\langle -\frac{12}{17}, \frac{48}{17} \right\rangle$$

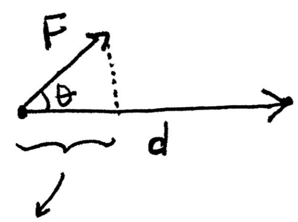
$$\text{scal}_u v = \frac{u \cdot v}{|u|} = \frac{12}{\sqrt{17}}$$

Applications of Dot Products.

Def Work.

Let a constant force $\mathbf{F}$ be applied to an object, producing a displacement $\mathbf{d}$. If the angle between $\mathbf{F}$ and $\mathbf{d}$ is θ, then the work done by the force is

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F}\cdot\mathbf{d}$$



$|F|\cos\theta$
only this component of F does work.

Ex A force $F = \langle 3, 3, 2\rangle$ (in newtons) moves an object along a line segment from $P(1,1,0)$ to $Q(6,6,0)$ (in meters). What is the ~~[illegible]~~ work done by the force?

$\overrightarrow{PQ} = \langle 5, 5, 0\rangle$

$\therefore\ W = F\cdot\overrightarrow{PQ} = (5)(3) + (5)(3) + (0)(2) = 30$ J.