12.4 The cross product of two vectors

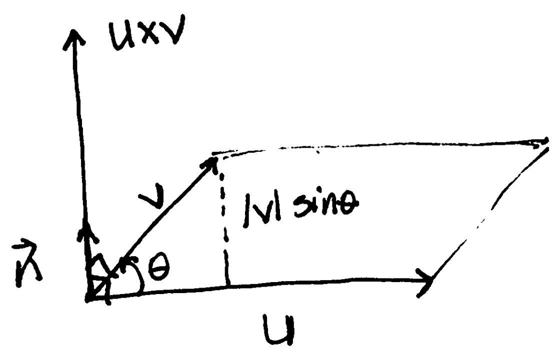
Many problems in geometry require us to find a vector that is perpendicular to each of two given vectors u and v. One way of doing this is provided by the cross product of u and v denoted by $u \times v$.

Lets define this new product

<u>Def</u> Given two non-zero vectors u and v in $\mathbb{R}^3$, the <u>cross product</u> $u \times v$ is defined as follows:

$$u \times v = |u||v| \sin\theta \, \vec{n}$$

where $0 \le \theta \le \pi$ is the angle between u and v and $\vec{n}$ is a unit vector which is normal (perpendicular) to u and v whose direction is determined by the right-hand thumb rule.



- u and v are parallel ($\theta = 0$ or $\theta = \pi$) if and only if $u \times v = 0$.

- If u and v are two sides of a parallelogram, then the area of this parallelogram is:

$$|u \times v| = |u|\,|v| \sin\theta$$

- $v \times u = -(u \times v)$

- $u \times v$ is orthogonal to u and v (by definition) that means $(u \times v) \cdot u = 0$ and $(u \times v) \cdot v = 0$

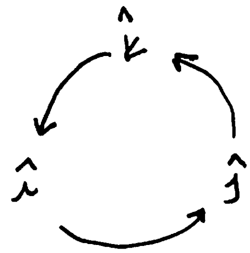
- Distributivity with respect to addition

$$u \times (v + w) = u \times v + u \times w$$
$$(u + v) \times w = u \times w + v \times w$$

- If c is a scalar,

$$(cu) \times v = c(u \times v) = u \times (cv)$$

Cross products of unit vectors



$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k} \qquad \hat{i} \times \hat{i} = 0$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad k \times \hat{j} = -\hat{i} \qquad \hat{j} \times \hat{j} = 0$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j} \qquad \hat{k} \times \hat{k} = 0$$

Let $u = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$; $v = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

$$\Rightarrow u \times v = (u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}) \times (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k})$$

$$= (u_1 v_1)(\hat{i} \times \hat{i}) + u_1 v_2 (\hat{i} \times \hat{j}) + \cdots + (u_3 v_3) \hat{k} \times \hat{k}$$

$$\vdots$$

$$= (u_2 v_3 - u_3 v_2) \hat{i} - (u_1 v_3 - u_3 v_1) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}$$

<u>Thm</u>

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \longrightarrow \text{components of } u \\ \longrightarrow \text{components of } v \end{array}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Example Calculate the cross product of $u = 2\hat{i} - \hat{j} + 4\hat{k}$ and $v = \hat{i} + 5\hat{j} - 3\hat{k}$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & 4 \\ 5 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix}$$

$$= \hat{i}(3-20) - \hat{j}(-6-4) + \hat{k}(10+1)$$

$$= -17\hat{i} + 10\hat{j} + 11\hat{k}$$

$$(u \times v) \cdot u = -34 - 10 + 44 = 0$$

$$(u \times v) \cdot v = -17 + 50 - 33 = 0.$$

Example Find the area of the triangle whose vertices are $P(2,-1,3)$, $Q(1,2,4)$ and $R(3,1,1)$

$$\overrightarrow{PQ} = -\hat{i} + 3\hat{j} + \hat{k}$$

$$\overrightarrow{PR} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -8\hat{i} - \hat{j} - 5\hat{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = 3\sqrt{10}$$

$$\therefore \text{ area of triangle } = \frac{3\sqrt{10}}{2}$$