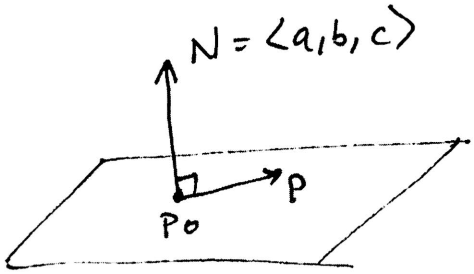
13.1 Planes and Surfaces

A plane can be characterized in several ways:

- As the plane through three noncollinear points
- As the plane through a line and a point not on the line
- As the plane through a point and perpendicular to a specified direction

The third approach is the most convenient for us.

Let $P_0(x_0, y_0, z_0)$ be a given point and $N = \langle a, b, c \rangle$ a direction vector perpendicular to the plane



Then the point $P(x, y, z)$ is on the plane $\Longleftrightarrow$ $\overrightarrow{P_0P}$ is perpendicular to N.

$$\boxed{\overrightarrow{P_0P} \cdot N = 0}$$ Vector equation of a plane

$\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$\Rightarrow$ $\overrightarrow{P_0P} \cdot N = 0$ can be written as follows:

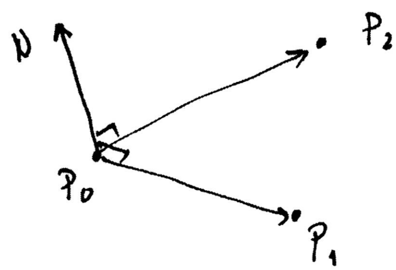
$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$\therefore$ $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$ Cartesian equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $N = \langle a, b, c \rangle$

The last equation can also be written as:

$$ax + by + cz = d \qquad \text{where } d = ax_0 + by_0 + cz_0.$$

Ex. Find an equation for the plane through the three points $P_0 = (3, 2, -1)$, $P_1 = (1, -1, 3)$ and $P_2 = (3, -2, 4)$



$\overrightarrow{P_0P_1} = \langle -2, -3, 4 \rangle$

$\overrightarrow{P_0P_2} = \langle 0, -4, 5 \rangle$

$$N = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -3 & 4 \\ 0 & -4 & 5 \end{vmatrix} = \hat{i} + 10\hat{j} + 8\hat{k}$$

We have the information that we need to give the equation of the plane.

we need a point: (we are given three, choose any) $P_0 = (3, 2, -1)$

we need a normal vector: $N = \langle 1, 10, 8 \rangle$

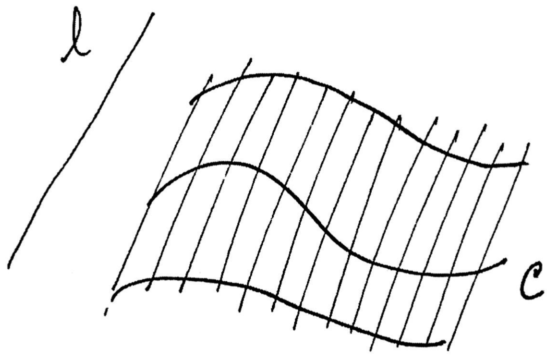
$\therefore$ Equation of the plane:

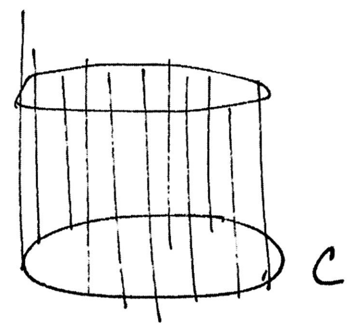
$$1(x - 3) + 10(y - 2) + 8(z + 1) = 0$$

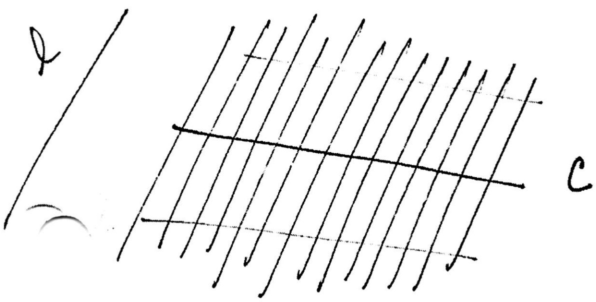
$$\boxed{x + 10y + 8z = 15}$$

Cylinders.

Def. Given a curve C in a plane P and a line l not in P, a cylinder is the surface consisting of all lines parallel to l that pass through C.





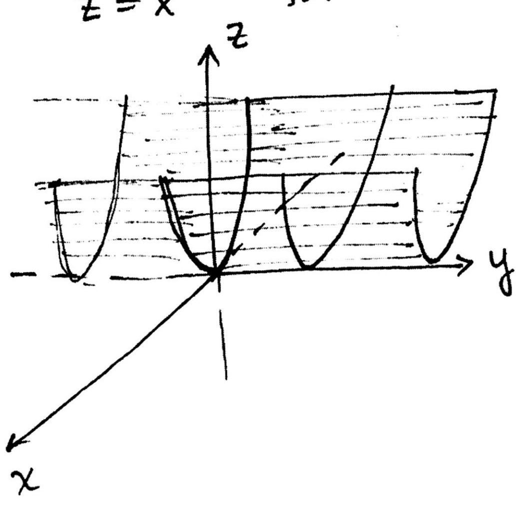


(a plane in particular is also a cylinder ☺).

Consider the equation $z = x^2$

this equation represents a parabola in the xz-plane that opens towards the positive z-axis, since y is not present, we interpret this equation as:

$z = x^2$ for all values of y. The graph will be:



this can be viewed as a cylinder with

C: $z = x^2$

l: y-axis

- Any equation in rectangular coordinates x, y, z with one variable missing represents a cylinder whose rulings are parallel to the axis corresponding to the missing variable.

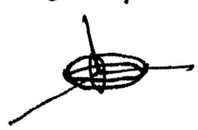
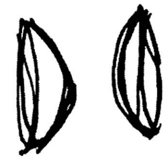
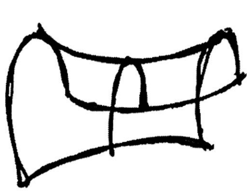
Quadric Surfaces

In three-dimensional space the most general equation of the second degree is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

The graph of such an equation is called a <u>quadric surface</u>.

There are exactly six distinct types of nondegenerate quadric surfaces:

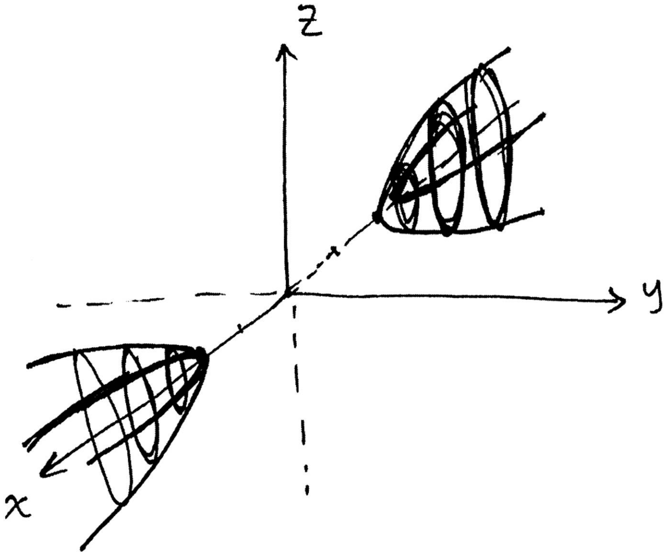
- The ellipsoid  $\rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- The hyperboloid of one sheet  $\rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- The hyperboloid of two sheets  $\rightarrow$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- The elliptic cone  $\rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
- The elliptic paraboloid  $z = ax^2 + by^2$
- The hyperbolic paraboloid  $z = by^2 - ax^2$

Ex Sketch and identify the surface:

$$x^2 - 4y^2 - z^2 = 4$$

This equation represents a hyperboloid of two sheets with the x-axis as its major axis.

- if $z=0$ $\Rightarrow$ $x^2 - 4y^2 = 4$ hyperbola in the xy-plane
- if $y=0$ $\Rightarrow$ $x^2 - z^2 = 4$ hyperbola in the xz-plane



- $x^2 - 4y^2 - z^2 = 4$ $\Rightarrow$ $4y^2 + z^2 = x^2 - 4$

if $x^2 - 4 \geq 0$, say, $x^2 - 4 = k^2$

then $4y^2 + z^2 = k^2$ represents an ellipse in the plane $x = \pm\sqrt{k^2+4}$

planes parallel to the yz plane.

Ex. $z = y^2 - x^2$

hyperbolic paraboloid.

if $x=0$ (plane yz), $z = y^2$ parabola

if $y=0$ (plane xz), $z = -x^2$ parabola.

if $z = k$

(a) $k>0$ $y^2-x^2=k$ hyperbola with y-axis as its major axis.

(b) $k<0$ $y^2-x^2=k$ hyperbola with x-axis as its major axis