12.5 Lines and Curves in Space

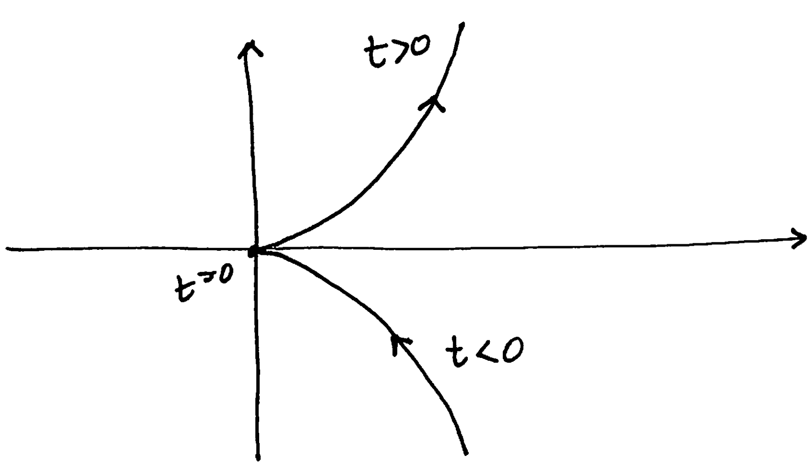
Parametric equations of curves.

In physical problems we often consider a moving point, and t is understood to be the time measured from the moment at which the motion begins.
The point P whose coordinates are x and y then traces out the curve as t traverses some definite interval, say $t_1 \le t \le t_2$.
This provides not only a description of the path on which the point moves, but also information about the direction of its motion and its location on the path for various values of t.

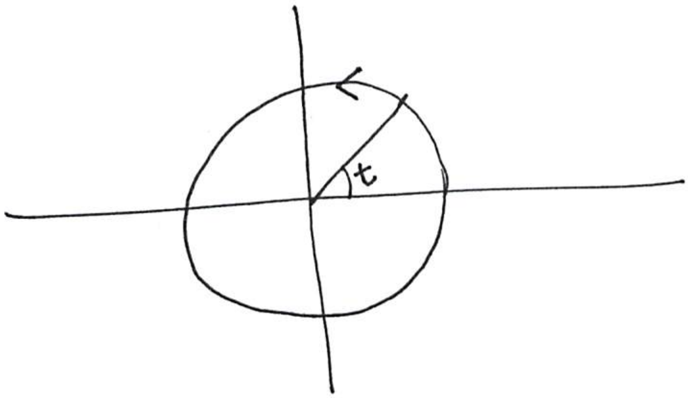
t parameter
x, y dependend variables

$$x = f(t), \quad \text{and} \quad y = g(t)$$

Ex 1 $\quad x = t^2, \quad y = t^3$



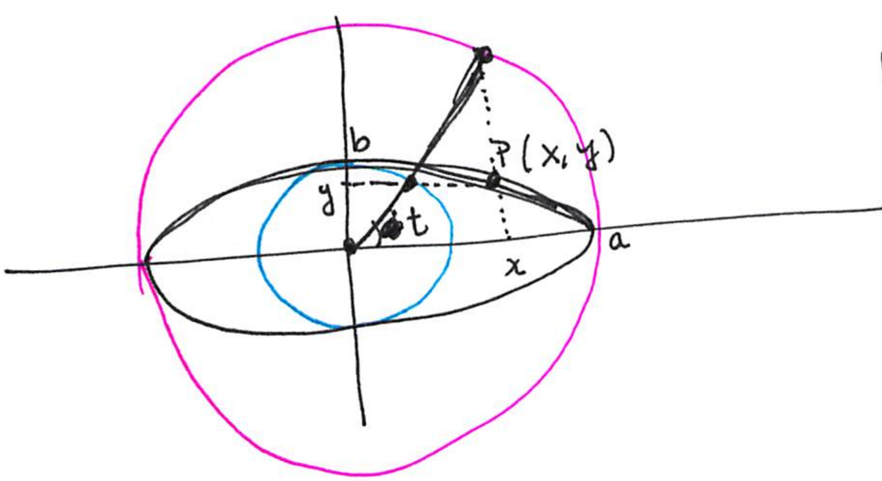
Ex 2 $x = a\cos t\,;\quad y = a\sin t \qquad 0 \le t \le 2\pi$



circle centered at the origin with radius a.

$$x^2 + y^2 = a^2$$

Ex 3 $x = a\cos t\;;\quad y = b\sin t$



Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

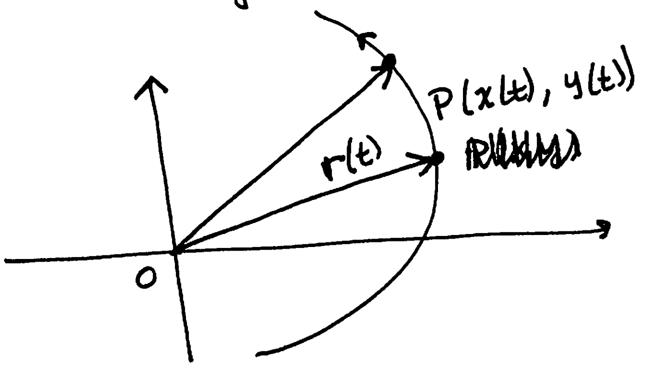
$$0 \le t \le 2\pi$$

Connection between vectors and parametric equations of curves.

Parametric equations of a curve ~~described by the motion of a point~~

$$x = x(t), \qquad y = y(t)$$

A more concise description of the motion is obtained by using the position vector of the moving point,



$\mathbf{r}(t) = \overrightarrow{OP} = \langle x(t), y(t) \rangle$

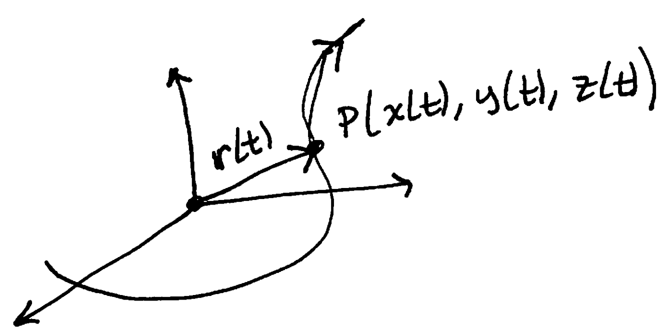
$\overrightarrow{OP} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

- $x = x(t), \quad y = y(t)$
- $\mathbf{r}(t) = \langle x(t), y(t) \rangle$

The study of ~~a pair of~~ parametric equations is equivalent to the study of a single vector function.

$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$

$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3 \qquad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$



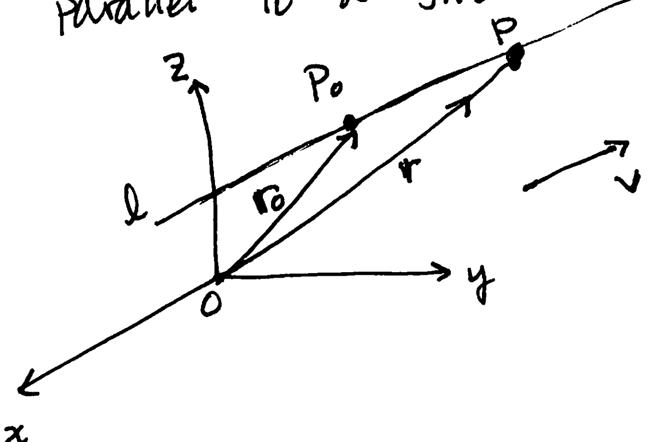
Lines in space.

A line in space can be given geometrically in three ways:

- As the line through two points
- As the intersection of two planes
- As the line through a point in a specified direction.

The third way is the most important for us

Suppose ℓ is the line in space that passes through a given point $P_0 = (x_0, y_0, z_0)$ and is parallel to a given non-zero vector $\mathbf{v} = \langle a, b, c \rangle$



Then another point $P = (x, y, z)$ lies on the line ℓ if and only if the vector $\overrightarrow{P_0P}$ is parallel to the vector $\mathbf{v}$. That is

(1) $\boxed{\overrightarrow{P_0P} = t\,\mathbf{v}}$ for some real number t

If $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$

and equation (1) becomes

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$$

$$\therefore \boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}}$$ Vector equation of the line l

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \\ & t \in \mathbb{R} \end{aligned}}$$ parametric equations of the line l.

<u>Ex</u> A line l goes through the points $P_0 = (3, -2, 1)$ and $P_1 = (5, 1, 0)$

(a) Find the parametric equations of l.

(b) Find the points at which this line pierces the three coordinate planes.

(a)

$x = 3 + 2t$

$y = -2 + 3t$

$z = 1 - t$

$t \in \mathbb{R}$

(b) xy-plane $(5, 1, 0)$

xz-plane $(\frac{13}{3}, 0, \frac{1}{3})$

yz-plane $(0, -\frac{13}{2}, \frac{5}{2})$

Curves in space $\quad \mathbf{r}: \mathbb{R} \to \mathbb{R}^3$

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

- The domain of $\mathbf{r}$ is the largest set of values of t on which all of f, g, and h are defined.

- Just as in ordinary calculus, $\mathbf{r}(t)$ is said to be continuous at $t = t_0$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

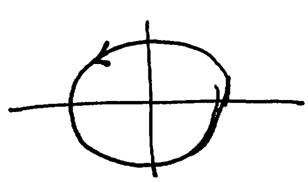
which means that $|\mathbf{r}(t) - \mathbf{r}(t_0)|$ can be made as small as we please by taking t sufficiently close to t_0.

$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous $\iff$ $x(t)$, $y(t)$ and $z(t)$ are continuous.

Orientation of curves. The positive orientation is the direction at which the curve is generated as the parameter increases from a to b.

Ex $\quad \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$



The orientation of a parameterized curve and its tangent vectors are consistent: The positive orientation of the curve is the direction in which the tangent vectors point along the curve.

<u>Ex.</u> Helix $\mathbf{r}(t) = \langle 4\cos t, \sin t, \frac{t}{2\pi} \rangle$.

<u>Ex</u> $\mathbf{r}(t) = \cos\pi t\, \hat{\imath} + \sin\pi t\, \hat{\jmath} + e^{-t}\hat{k}$ $\qquad t \geq 0$

(a) Evaluate $\lim_{t \to 2} \mathbf{r}(t)$

(b) Evaluate $\lim_{t \to \infty} \mathbf{r}(t)$

(c) At what points is $\mathbf{r}$ continuous.