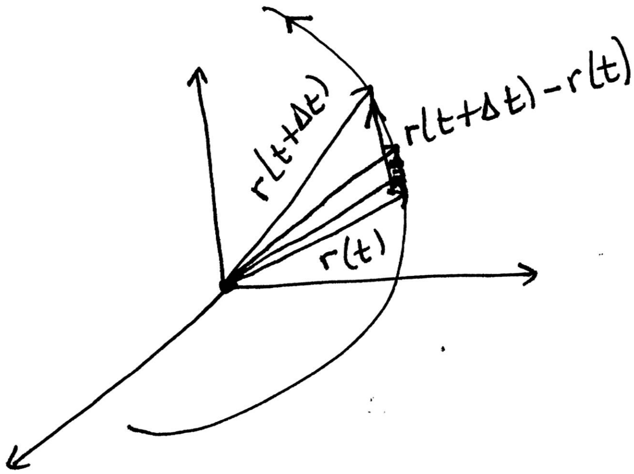
12.6 Calculus of vector-valued functions.



$\Delta t \to 0$

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t) \rangle - \langle f(t), g(t), h(t) \rangle}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle.$$

<u>Def</u> Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions on (a,b). Then $\mathbf{r}$ is differentiable on (a,b) and

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Provided $\mathbf{r}'(t) \neq \vec{0}$, $\mathbf{r}'(t)$ is a tangent vector at the point corresponding to $\mathbf{r}(t)$.

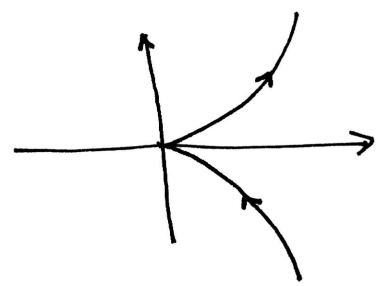
- The vector $\mathbf{r}'(t)$ points in the direction of the curve at P.

- $\mathbf{r}'(t)$ gives the rate of change of the function $\mathbf{r}(t)$ at the point P.

- If $\mathbf{r}(t)$ is the position function of a moving object, then $\mathbf{r}'(t)$ is the velocity vector of the object, which always points in the direction of motion, and $|\mathbf{r}'(t)|$ is the speed of the object.

→ $x(t) = t^2$
$y(t) = t^3$

$\mathbf{r}(t) = \langle t^2, t^3 \rangle$



This curve has a cusp at the origin.

$\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle$

$\mathbf{r}'(0) = \langle 0, 0 \rangle$

$\mathbf{r}'(0) = \vec{0}$

$\mathbf{r}'(0) = \vec{0}$ means the velocity is zero at $t=0$, At such a stationary point, the object may change direction abruptly, creating a cusp in its trajectory.

- <u>Def</u> $\mathbf{r}(t)$ is <u>smooth</u> on an interval if $\mathbf{r}(t)$ is differentiable and $\mathbf{r}'(t) \neq \vec{0}$ on that interval. Smooth curves have no cusps or corners.

Def. Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a smooth parameterized curve, for $a \le t \le b$

The unit tangent vector for a particular value of t is:

$$T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Example. Find the unit tangent vector for the following parameterized curve:

$$\mathbf{r}(t) = \langle t^2, 4t, 4\ln t \rangle \qquad t > 0$$

$$\mathbf{r}'(t) = \langle 2t, 4, \frac{4}{t} \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 16 + \frac{16}{t^2}} = \sqrt{\left(2t + \frac{4}{t}\right)^2}$$

$$= \left|2t + \frac{4}{t}\right| = 2t + \frac{4}{t} \quad (\text{since } t > 0)$$

$$\therefore\ T(t) = \left\langle \frac{t^2}{2t + \frac{4}{t}}, \frac{4t}{2t + \frac{4}{t}}, \frac{4\ln t}{2t + \frac{4}{t}} \right\rangle.$$

Thm Derivative rules.

Let u and v be differentiable vector-valued functions and let f be a differentiable scalar-valued function, all at a point t. Let $\vec{c}$ be a constant vector. The following rules apply.

1. $\frac{d}{dt}(\vec{c}) = \vec{0}$ constant rule

2. $\frac{d}{dt}\left(u(t) + v(t)\right) = u'(t) + v'(t)$ sum rule

3. $\frac{d}{dt}\left(f(t)\, u(t)\right) = f'(t)\, u(t) + f(t)\, u'(t)$ Product rule

4. $\frac{d}{dt}\left(u(f(t))\right) = u'(f(t))\, f'(t)$ Chain rule

5. $\frac{d}{dt}\left(u(t) \cdot v(t)\right) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$ Dot Product Rule

6. $\frac{d}{dt}\left(u(t) \times v(t)\right) = u'(t) \times v(t) + u(t) \times v'(t)$ Cross product rule.

Integrals of vector-valued functions.

<u>Def.</u> Let $r(t) = \langle f(t), g(t), h(t) \rangle$ be a vector-function and let $R(t) = \langle F(t), G(t), H(t) \rangle$ where F, G, and H are antiderivatives of f, g, and h, respectively. The indefinite integral of r is

$$\int r(t)\,dt = R(t) + \vec{c}$$

where $\vec{c}$ is an arbitrary constant vector.

In component form:

$$\int \langle f(t), g(t), h(t) \rangle\,dt = \langle F(t), G(t), H(t) \rangle + \langle c_1, c_2, c_3 \rangle.$$

<u>Ex.</u> $$\int (\hat{i} + 2t\hat{j} + 3t^2\hat{k})\,dt = (t\,\hat{i} + t^2\hat{j} + t^3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= (t + c_1)\hat{i} + (t^2 + c_2)\hat{j} + (t^3 + c_3)\hat{k}.$$

<u>Def</u>. Definite integral of a vector-valued function.

Let $\mathbf{r}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$, where f, g, and h are integrable on the interval $[a,b]$.

The definite integral of $\mathbf{r}$ on $[a,b]$ is:

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{\imath} + \left(\int_a^b g(t)\,dt\right)\hat{\jmath} + \left(\int_a^b h(t)\,dt\right)\hat{k}.$$