12.7 Motion in Space

Def. Position, velocity, speed, acceleration

Let the position of an object moving in three-dimensional space be given by:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad \text{for } t \geq 0$$

The velocity of the object is: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

The speed of the object is: $|\mathbf{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

The acceleration of the object is: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

Thm Let $\mathbf{r}$ describe a path on which $|\mathbf{r}|$ is constant (motion on a circle or sphere centered at the origin), then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the position vector and the velocity vector are orthogonal at all times.

Pf. $|\mathbf{r}(t)| = c \Rightarrow |\mathbf{r}(t)|^2 = c^2$

$\Rightarrow \mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$

now, lets differentiate both sides of this equation with respect to t:

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

$$\Rightarrow 2\,\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

$$\therefore \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \qquad \therefore \mathbf{r}(t) \cdot \mathbf{v}(t) = 0$$

//.

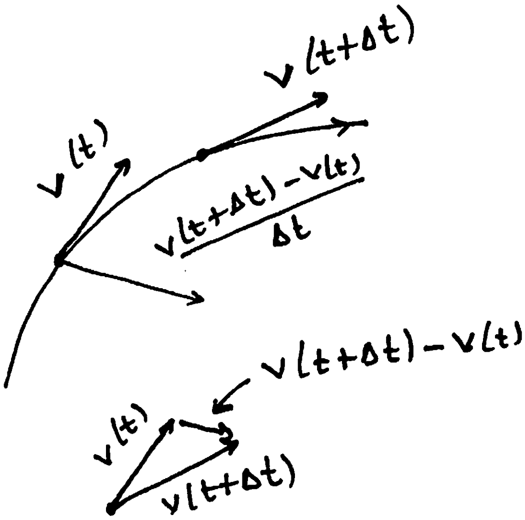
Observations:

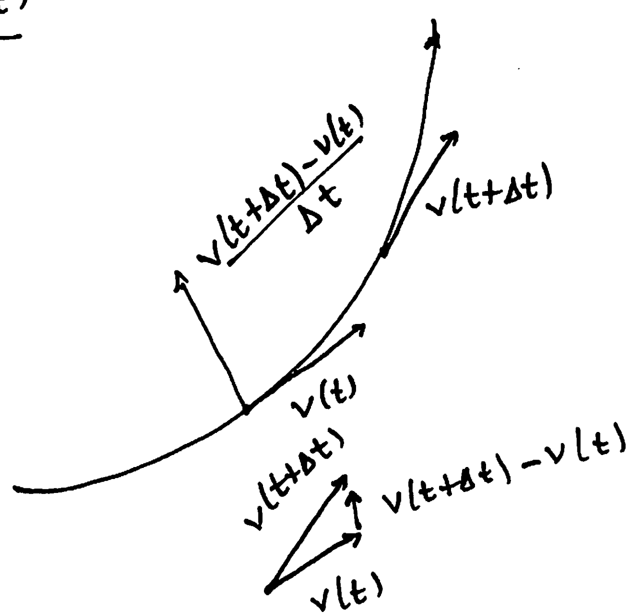
- $r(t)$ represents the position vector at t

  $v(t) = r'(t)$ is a tangent vector to the curve at t that points in the direction of the moving object and whose length is the speed of the object.

  $a(t) = r''(t)$ is the acceleration vector, usually points toward the concave side of the curve

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$





Example. Path on a sphere. An object moves on a trajectory described by

$$\mathbf{r}(t) = \langle 3\cos t,\ 5\sin t,\ 4\cos t \rangle \qquad 0 \le t \le 2\pi$$

(a) Show that the object moves on a sphere and find the radius of the sphere.

$$x^2 + y^2 + z^2 = 9\cos^2 t + 25\sin^2 t + 16\cos^2 t$$

$$= 25(\cos^2 t + \sin^2 t) = 25$$

$\therefore$ The object moves along a sphere of radius 5 centered at the origin.

(b) Find the velocity and speed of the object

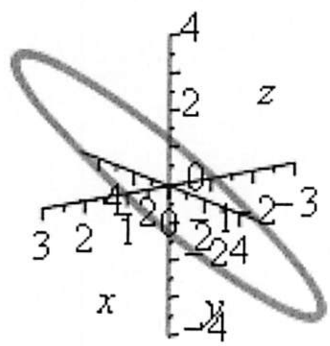
- $\mathbf{v}(t) = \langle -3\sin t,\ 5\cos t,\ -4\sin t \rangle$ velocity vector
- speed $= |\mathbf{v}(t)| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t}$
  $= 5.$

The speed of the object is always 5.

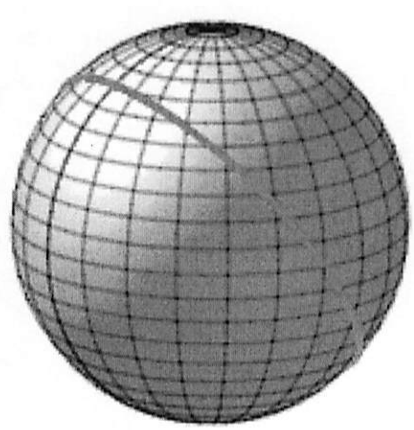
Observe that $\mathbf{r} \cdot \mathbf{v} = 0$ (circular motion).

Graph of the trajectory r(t)=< 3 cos(t), 5 sin(t), 4 cos(t) > $0 \le t \le 2\pi$
The graph is a circle in the space that lies on a sphere centered at the origin of radius 5

```
> with(plottools) :
> with(plots) :
> c := sphere([0, 0, 0], 5, scaling = constrained, axes = boxed, labels = [x, y, z]) :
> display(c) :
> d := spacecurve([3·cos(t), 5·sin(t), 4·cos(t)], t = 0..2*Pi, axes = normal, labels = [x, y, z],
      color = red, thickness = 3) :
> display(d)
```



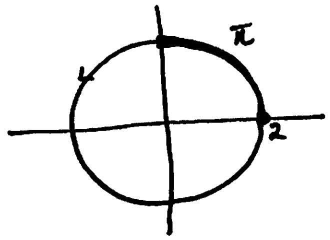

```
> display(c, d)
```

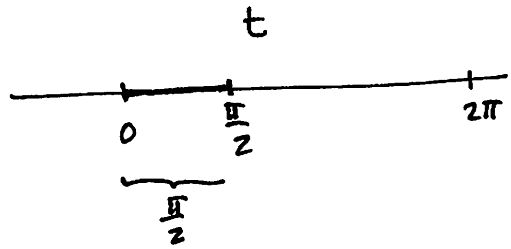


```
>
```

Ex Consider:

$r(t) = 2\cos t\ \hat{i} + 2\sin t\ \hat{j}$ $0 \le t \le 2\pi$



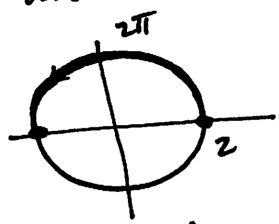


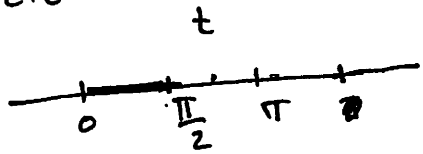
total length of circle: $2\pi(2) = 4\pi$

In this parametrization, the parameter has swept a length of $\frac{\pi}{2}$ units and the object on the curve has swept a length of π units.

now, if we take $r(t) = 2\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j}$, $0 \le t \le \pi$

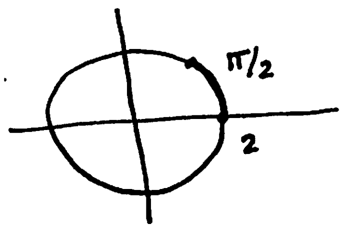
this also describes a circle of radius 2.

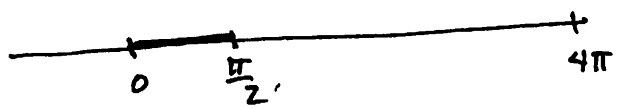




and when t sweeps a length of $\frac{\pi}{2}$ units, the object sweep a length of 2π units

Finally consider $r(t) = 2\cos\left(\frac{t}{2}\right)\hat{i} + 2\sin\left(\frac{t}{2}\right)\hat{j}$ $0 \le t \le 4\pi$





t sweeps a length of $\frac{\pi}{2}$ units and the object also sweeps a length of $\frac{\pi}{2}$ units

This parametrizations like the last one are of importance for us. They are called arc-length parametrizations.