12.8 Lengths of curves

<u>Def.</u> Arc Length for Vector Functions

Consider the parameterized curve $r(t) = \langle x(t), y(t), z(t) \rangle$, where $x'(t)$, $y'(t)$, and $z'(t)$ are continuous, and the curve is traversed once for $a \le t \le b$. The arc length of the curve is:

$$L = \int_a^b |v(t)|\, dt$$

Now, lets make the arc length variable,

Let $s(t) = \int_a^t |v(u)|\, du$, so we have the arc length function, $s(t)$ is the length of the curve from a to t.

If $s(t) = \int_a^t |v(u)|\, du$, then by the fundamental theorem of calculus $\frac{ds}{dt} = |v(t)|$

The arc length function gives the relationship between the arc length of a curve and any parameter t used to describe the curve.

- A curve is parameterized by arc length if and only if $|v(t)| = 1$

<u>Pf.</u> $\Rightarrow)$ Suppose a curve is parameterized by arc length

$$\Rightarrow \underbrace{s(t) = \int_a^t |v(u)|\,du}_{\text{length of curve}} = \underbrace{t-a}_{\text{length of interval}} \quad \forall \; t \geq a$$

$$\Rightarrow \quad s(t) = t - a$$

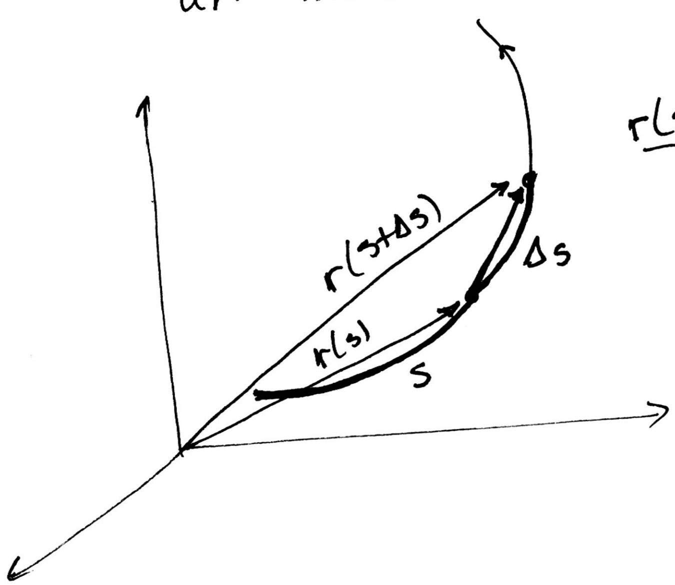
$$\therefore \quad \frac{ds}{dt} = |v(t)| = 1$$

$\Leftarrow)$ Now, suppose $|v(t)| = 1 \quad \forall t$

$$\Rightarrow \quad s(t) = \int_a^t |v(u)|\,du = \int_a^t du = u\Big|_a^t = t - a$$

$\Rightarrow$ length of the curve from a to t is equal to the length swept by the parameter from a to t.

- An increment of Δs in the parameter corresponds to an increment of exactly Δs in the arc length.



$$\frac{r(s+\Delta s) - r(s)}{\Delta s}$$

notice that the length of the vector $r(s+\Delta s) - r(s)$ is close to the length Δs, so, geometrically, the vector $\frac{r(s+\Delta s) - r(s)}{\Delta s}$ has length close to 1 and as $\Delta s \to 0$, the length of $\frac{dr}{ds}$ will be 1.

Ex. Determine whether the following curve uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

$$\mathbf{r}(t) = \langle e^t, e^t, e^t \rangle, \qquad t \geq 0$$

$$\mathbf{v}(t) = \langle e^t, e^t, e^t \rangle$$

$$\Rightarrow |\mathbf{v}(t)| = \sqrt{(e^t)^2 + (e^t)^2 + (e^t)^2} = \sqrt{3e^{2t}} = \sqrt{3}\, e^t \neq 1$$

$\therefore$ the curve is not parameterized by arc length.

Now, to reparameterize with arc length we find $s(t)$:

$$s(t) = \int_0^t \sqrt{3}\, e^u \, du = \sqrt{3}\, e^u \Big|_0^t = \sqrt{3}\,(e^t - 1)$$

$$\Rightarrow \quad s = \sqrt{3}\,(e^t - 1)$$

$$\Rightarrow \quad \frac{s}{\sqrt{3}} + 1 = e^t \qquad \therefore\ t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

writing the curve with the parameter s, we get:

$$\mathbf{r}(s) = \left\langle \frac{s}{\sqrt{3}} + 1,\ \frac{s}{\sqrt{3}} + 1,\ \frac{s}{\sqrt{3}} + 1 \right\rangle$$

This description has the property that an increment of Δs in the parameter corresponds to an increment of exactly Δs in the arc length.