12.9 Curvature and Normal Vectors

In this section we are going to explore the information that we can obtain about a curve when we parameterize with arc length. Then using the chain rule we will be able to find all this information from any curve parameterized with any parameter and with no need to reparameterite the curve with the arc length parameter.

First, lets assume that we are given a vector function parameterized with arc length:

$\mathbf{r}(s)$

The derivative is $\frac{d\mathbf{r}}{ds}$

if t is the parameter time, by the chain rule:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = \frac{d\mathbf{r}}{ds} |\mathbf{r}'(t)|$$

$$\therefore \mathbf{r}'(t) = \frac{d\mathbf{r}}{ds} |\mathbf{r}'(t)|$$

$$\therefore \frac{d\mathbf{r}}{ds} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Remember:

$$s(t) = \int_a^t |\mathbf{v}(u)|\, du$$

$$\therefore \frac{ds}{dt} = |\mathbf{v}(t)| = |\mathbf{r}'(t)|$$

$\therefore \frac{d\mathbf{r}}{ds}$ represents a <u>tangent vector</u> of length 1, we call this vector T

$$\boxed{T = \frac{d\mathbf{r}}{ds}}$$

Now, lets analyze $\frac{dT}{ds}$, in other words $\frac{d^2\mathbf{r}}{ds^2}$

we will be using the following theorem in this section:

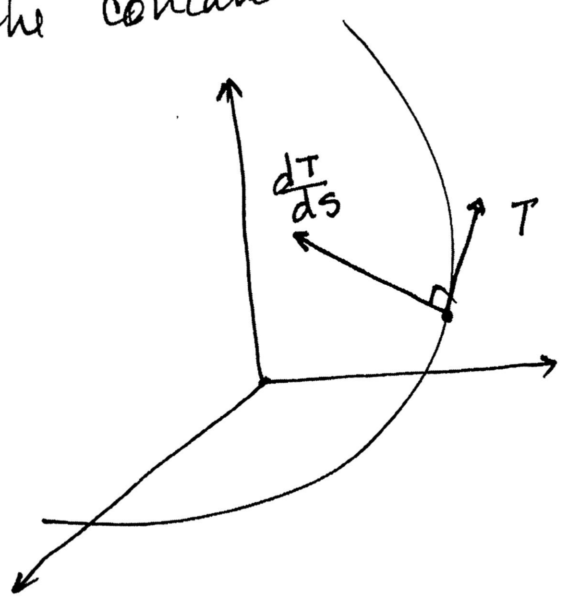
Recall: If the magnitude of a vector function is constant for all values, then the vector function and its derivative are orthogonal.

$$|\mathbf{r}| = c \Rightarrow \mathbf{r} \cdot \mathbf{r}' = 0$$

Since $|T| = 1$ $\forall s$, then $T \cdot \frac{dT}{ds} = 0$

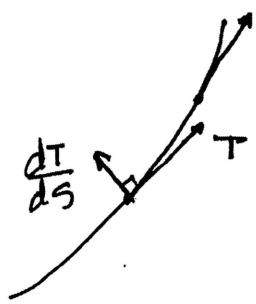
$\therefore$ $\frac{dT}{ds}$ is a vector function that is orthogonal to the unit tangent vector function T $\forall s$.

Previously, we saw a geometric argument that the second derivative vector usually points toward the concave side of the curve.



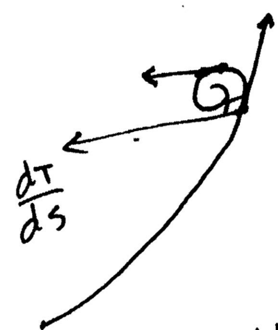
What does $\frac{dT}{ds}$ measure ?

A vector changes in magnitude and direction, since T is always a unit vector, then $\frac{dT}{ds}$ measures only how much is the direction of T changing.



In this example the direction of T is not changing much

$\left|\frac{dT}{ds}\right|$ small



In this example the direction of T has a big change

$\left|\frac{dT}{ds}\right|$ big

The magnitude of $\frac{dT}{ds}$ measures how much the curve is "curving",

we call $\left|\frac{dT}{ds}\right|$ the curvature of the curve, and

we denote it by the letter κ

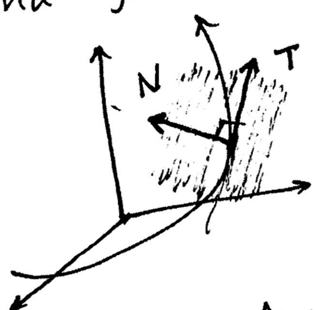
<u>Def.</u> The <u>curvature</u> of the vector function $r(s)$ is defined as follows:

$$\boxed{\kappa = \left|\frac{dT}{ds}\right|}$$

Next, we define the <u>normal vector</u>, this is just the vector $\frac{dT}{ds}$ divided by its magnitude:

$$\boxed{N = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|}} \quad , \quad \text{equivalently}, \quad \boxed{N = \frac{\frac{dT}{ds}}{\kappa}}$$

N is a unit vector that is orthogonal to T and goes in the same direction as the vector $\frac{dT}{ds}$



T and N define a plane, this is called the <u>osculating</u> plane.

The next question is: how quickly does the curve moves out of the plane determined by T and N ?

To answer this we will construct a vector orthogonal to the plane and analyze how this vector changes. To construct this vector we take the cross product of T and N.

<u>Def.</u> The <u>binormal</u> vector is defined as:

$$\boxed{B = T \times N}$$

$|B| = 1$ since $|T| = 1$ and $|N| = 1$ and $\theta = \pi/2$

Now, lets see how B changes:

$$\frac{dB}{ds} = \frac{d}{ds}(T \times N) = \underbrace{\left(\frac{dT}{ds} \times N\right)}_{=0} + \left(T \times \frac{dN}{ds}\right)$$

$\frac{dT}{ds}$ and N are parallel vectors $\left(\frac{dT}{ds} = \kappa N \right)$

$$\therefore \frac{dT}{ds} \times N = 0$$

$$\therefore \frac{dB}{ds} = T \times \frac{dN}{ds}$$

$\frac{dB}{ds}$ measures how much the curve is twisting out of a plane.

Observations:

$\frac{dB}{ds}$ is orthogonal to T ($\frac{dB}{ds}$ is the crossproduct of T and $\frac{dN}{ds}$)

$\frac{dB}{ds}$ is orthogonal to B (since $|B| = 1$, $\forall s$ $B \cdot \frac{dB}{ds} = 0$

$\therefore \frac{dB}{ds}$ is orthogonal to T and B

$\therefore \frac{dB}{ds}$ is parallel to N

$$\therefore \frac{dB}{ds} = -\tau N$$

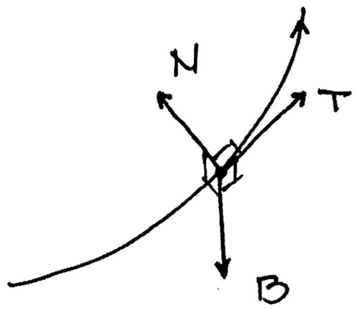
τ is called the torsion
(the negative sign is conventional)

we want an expression for τ, so lets take the dot product with N

$$\frac{dB}{ds} \cdot N = -\tau \underbrace{N \cdot N}_{=1}$$

$$\therefore -\tau = \frac{dB}{ds} \cdot N \; ; \; \boxed{\tau = -\frac{dB}{ds} \cdot N}$$

So, we have constructed a set of coordinate vector TNB for each point on the curve.



Local set of coordinates for the curve (Frenet - Serret frame)

- The plane defined by T and N is the osculating plane
- The plane defined by B and N is the normal plane
- T : gives information about the direction the curve is going.
- N : tells us the direction the curve turns within the osculating plane
- $\frac{dB}{ds}$: gives the rate at which the curve moves out of the osculating plane
- Curvature measures how much the curve has turned within the osculating plane. $\kappa = \left|\frac{dT}{ds}\right|$
- Torsion measures how much the curve has moved out of the osculating plane. $\tau = -\frac{dB}{ds} \cdot N$
- $B = T \times N$ binormal vector

Now, we will study how we can obtain all the information that we have when a curve is parameterized by arc length, if the curve is given in terms of any parameter t, without having to reparameterize the curve in terms of s.

Curve given: $\mathbf{r}(t)$ $\qquad$ t any parameter.

The key is to use the following relationship between s and t, together with the chain rule.

- $s(t) = \int_a^t |v(u)|\,du$

by the Fundamental Thm of Calculus:

$$\frac{ds}{dt} = |v(t)| \qquad \text{where } v(t) = \mathbf{r}'(t)$$

$$\therefore \underline{\frac{ds}{dt} = |\mathbf{r}'(t)|} \qquad (*)$$

1. What is $T = \frac{d\mathbf{r}}{ds}$ if we are given $\mathbf{r}(t)$

$$T = \frac{d\mathbf{r}}{ds} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$(*)$

chain rule

$$\therefore \boxed{T = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$$

2. What is $\dfrac{d^2 r}{ds^2}$?

$$T = \frac{dr}{ds}, \quad \Rightarrow \quad \frac{dT}{ds} = \frac{d^2 r}{ds^2}$$

$$\frac{dT}{ds} \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \frac{\frac{dT}{dt}}{\frac{ds}{dt}} \underset{(*)}{=} \frac{T'(t)}{|r'(t)|}$$

$$\therefore \boxed{\frac{dT}{ds} = \frac{T'(t)}{|r'(t)|}}$$

3. What is N ?

$$N = \frac{\frac{dT}{ds}}{\left|\frac{dT}{ds}\right|} \underset{\substack{\uparrow \\ \text{by part 2.}}}{=} \frac{\frac{T'(t)}{|r'(t)|}}{\left|\frac{T'(t)}{|r'(t)|}\right|} = \frac{\frac{T'(t)}{|r'(t)|}}{\frac{|T'(t)|}{|r'(t)|}} = \frac{T'(t)}{|T'(t)|}$$

$$\therefore \boxed{N = \frac{T'(t)}{|T'(t)|}}$$

4. What is κ ?

$$\kappa = \left|\frac{dT}{ds}\right| \underset{\substack{\uparrow \\ \text{by 2.}}}{=} \left|\frac{T'(t)}{|r'(t)|}\right| = \frac{|T'(t)|}{|r'(t)|}$$

$$\therefore \boxed{\kappa = \frac{|T'(t)|}{|r'(t)|}}$$

5. What is B ?

$$T = \frac{r'(t)}{|r'(t)|} \quad , \quad N = \frac{T'(t)}{|T'(t)|}$$

$$B = T \times N \quad ,$$

6. What is $\frac{dB}{ds}$?

$$\frac{dB}{ds} = \frac{\frac{dB}{dt}}{\frac{ds}{dt}} = \frac{\frac{dB}{dt}}{|r'(t)|}$$

chain rule

$$\therefore \boxed{\frac{dB}{ds} = \frac{B'(t)}{|r'(t)|}}$$

7. What is τ ?

$$\tau = -\frac{dB}{ds} \cdot N = -\frac{B'(t)}{|r'(t)|} \cdot \frac{T'(t)}{|T'(t)|}$$