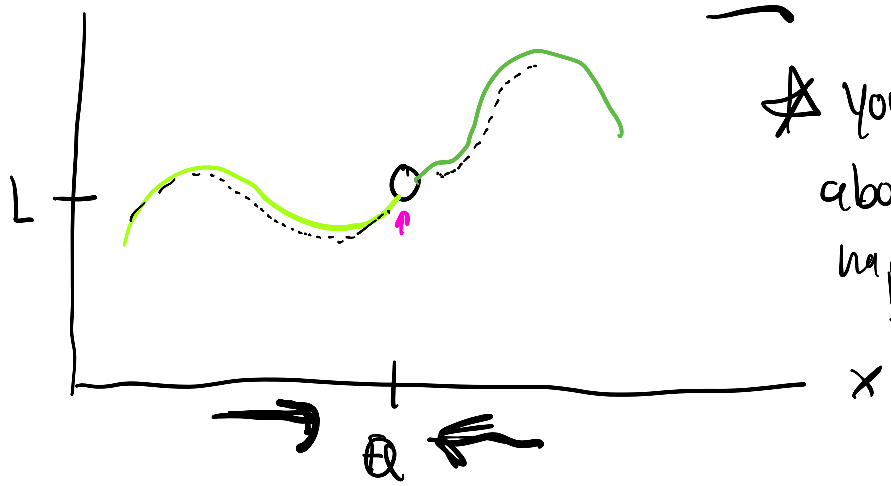
15.2 Limits and continuity (learn 2 path test for Nonexist. of Limits)

* we will exclude the ε-δ definitions and the concepts of open and closed sets from this chapter.
[This will be covered if you move onto Math 4547 Intro Analysis 1] *

Recall Thm 2.1 from Calc I (pg. 66 of this book)
Assume f is defined for all x near a (except possibly at a). Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} = L$.



* you don't care about what actually happens at f(a)!

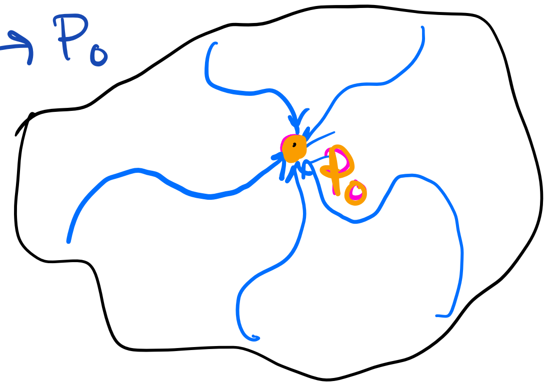
A function of 2 variables has a limit L as P(x,y) approaches a fixed point $P_0(a,b)$ if $|f(x,y) - L|$ can be made arbitrarily small for all P in the domain that are sufficiently close to P_0.

If such a limit exists, we write

If such a limit exists,

$$\lim_{(x,y)\to(a,b)} F(x,y) = \lim_{P\to P_0} f(x,y) = L$$

$F(a,b)$



One way to see the limit of f doesn't exist at a point (a,b) is to find two paths to P_0 in the domain and show that $f(x,y)$ values along those two paths approach different values as $(x,y)\to(a,b)$. This is called the two path test for nonexistence of limits and we will come back to this towards the end of the video.

Thm 15.2 [only tools needed are limit Laws analogous to those given in Thm 2.3]

[chapter 2 from Calc I.]

Let L and M be real #'s and suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is a constant, and $n > 0$ is an integer.

1. Sum $\lim_{(x,y)\to(a,b)} \left(f(x,y) + g(x,y)\right) = L + M$

2. Difference $\lim_{(x,y)\to(a,b)} \left(f(x,y) - g(x,y)\right) = L - M$

3. constant multiple $\lim_{(x,y)\to(a,b)} c\,f(x,y) = cL$

4. Product $\lim_{(x,y)\to(a,b)} f(x,y)\,g(x,y) = L \cdot M$

5. Quotient * $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ (only if $M \neq 0$)

6. Power $\lim \left(f(x,y)\right)^n = L^n$

$(x,y)\to(a,b)$

7. Root $\lim\limits_{(x,y)\to(a,b)} \left(f(x,y)\right)^{1/n} = L^{1/n}$

(Assume $L>0$) if n is even

Example:

Evaluate $\lim\limits_{(x,y)\to(2,8)} 3x^2y+\sqrt{xy}$

Step1: $= \lim\limits_{(x,y)\to(2,8)} 3x^2y + \lim\limits_{(x,y)\to(2,8)} \sqrt{xy}$ $\quad \sqrt{xy} = (xy)^{\frac{1}{2}}$

By 1.

Step2: $= 3\lim\limits_{(x,y)\to(2,8)} x^2y + \lim\limits_{(x,y)\to(2,8)} \sqrt{xy}$

By 3

Step3: $= 3\lim\limits_{(x,y)\to(2,8)} (x^2\cdot y) + \left(\lim\limits_{(x,y)\to(2,8)} (x\cdot y)\right)^{1/2}$

By 7.

Step4: $= 3\left(\lim\limits_{(x,y)\to(2,8)} x^2\right)\cdot\left(\lim\limits_{(x,y)\to(2,8)} y\right) +$

$\left[\left(\lim\limits_{(x,y)\to(2,8)} x\right)\cdot\left(\lim\limits_{(x,y)\to(2,8)} y\right)\right]^{1/2}$

By 4.

Step5: 2 8

By ⑥ $3\left(\lim_{(x,y)\to(2,8)} x\right)^2 \cdot \left(\lim_{(x,y)\to(2,8)} y\right) +$

$$\left[\left(\underbrace{\lim_{(x,y)\to(2,8)} x}_{2}\right)\cdot\left(\underbrace{\lim_{(x,y)\to(2,8)} y}_{8}\right)\right]^{1/2}$$

Step 4: Evaluate

$$= 3(2)^2(8) + \left((2)(8)\right)^{1/2}$$

$$= 3\cdot 4\cdot 8 + \sqrt{16}$$

$$= 96 + 4$$

$$= 100$$

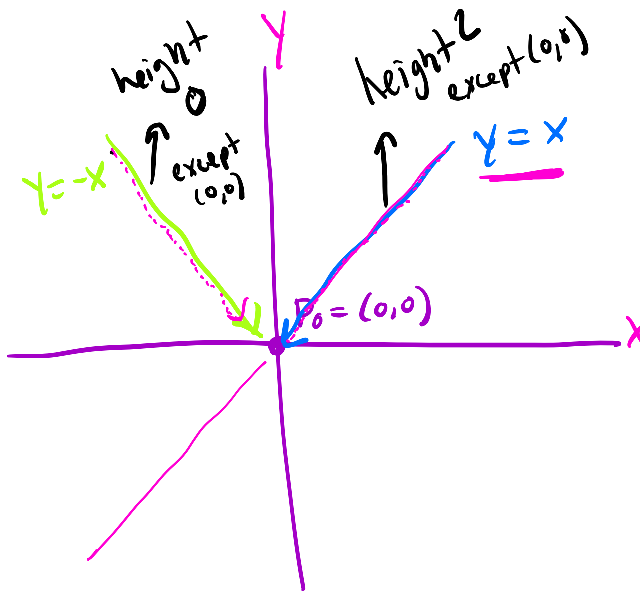
Two Path Test for Non existence of Limit

If $f(x,y)$ approaches two different values as (x,y) approaches (a,b) along two different paths in the domain of f, then $\lim f(x,y)$ doesn't or +, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ doesn't exist.

pg. 935

Ex: Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$

* Note we can't use Rule 5 from above because if $g(x,y) = x^2+y^2$, $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$.



If we look what points along $y = x$ (not at 0,0) evaluate to, we can sub in an x everywhere there is a y.

$$\frac{(x+y)^2}{x^2+y^2} \to \boxed{\frac{(x+x)^2}{x^2+x^2}} =$$

$$\frac{(2x)^2}{2x^2} = \frac{4x^2}{2x^2} = \boxed{2} \qquad x \neq 0$$

If we look along $y = -x$

$$\boxed{\frac{(x+y)^2}{\ \ }} \to \frac{(x+(-x))^2}{\ \ } \to \frac{0^2}{\ \ } \to \frac{0}{2x^2}$$

x^2+y^2 $\quad x^2+(-x)^2$, $\quad x^2+x^2$

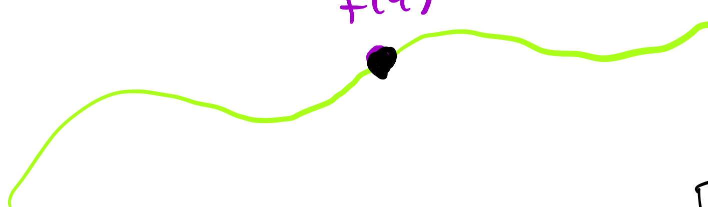
$=0$ $\quad$ if $x \neq 0$.

2 Because the function approaches 2 different values along 2 paths going to → (0,0), the limit does not exist at (0,0).

Calc 1 Continuity (pg. 104)

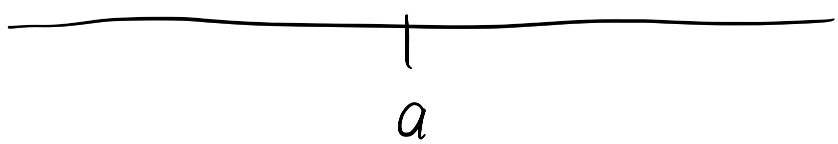
In order for f to be continuous at a, the three conditions must hold:

1. f(a) is defined
2. $\lim_{x\to a} f(a)$ exists
3. $\lim_{x\to a} f(x) = f(a)$



care about what happens at f(a).

$\mathbb{R} \to \mathbb{R}$



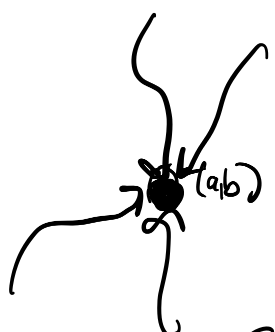
Continuity

Def: Continuity

The function f is continuous at the point (a,b) provided

1. f is defined at (a,b)
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, and
3. $\lim f(x,y) = f(a,b) = f(0,0) = 0$

$\mathbb{R}^2 \to \mathbb{R}$



$f(P) = f(a,b)$

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ ... $\lim_{P\to(a,b)}$

Ex: Is $f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

continuous at the point $(0,0)$?

Is f defined at $(0,0)$? Answer: Yes!

$\boxed{f(0,0) = 0}$

from definition.

$$\lim_{(x,y)\to(a,b)} f(x,y) = 0$$

If we can find at least one path where the limit of $f(x,y) \neq 0$ then
$(x,y)\to(0,0)$ along that path

we are done.

$x = y^2$ → $\dfrac{3xy^2}{x^2+y^4}$

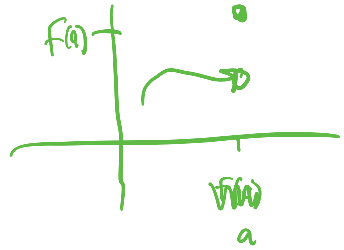
limit of $f(x,y)$ along path → $\dfrac{3(y^2)y^2}{(y^2)^2+y^4} = \dfrac{3y^4}{y^4+y^4} = \dfrac{3y^4}{2y^4} = \boxed{\dfrac{3}{2}}$

$\boxed{y^2 = x}$

We have at least one path whe

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$$

along path $y^2 = x$



So it is not continuous.