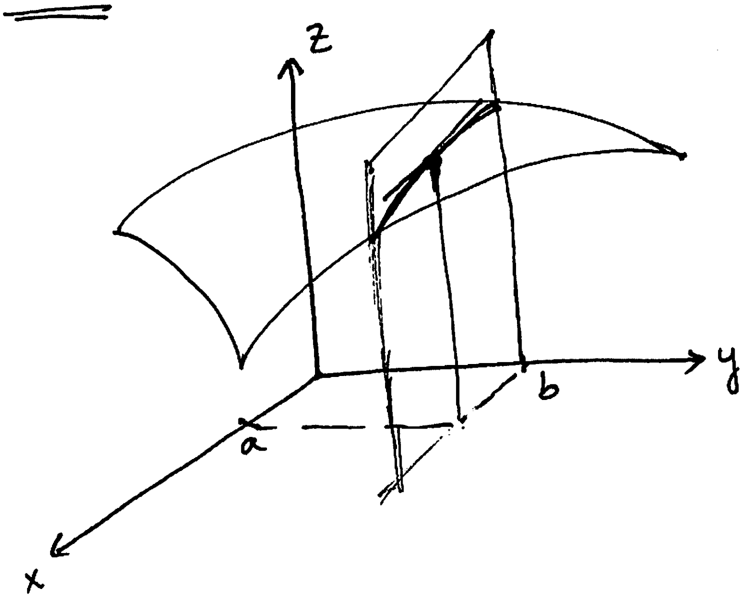
Continuity

Def. The function f is continuous at the point (a,b) if:

1. f is defined at (a,b)
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

13.4 Partial Derivatives.

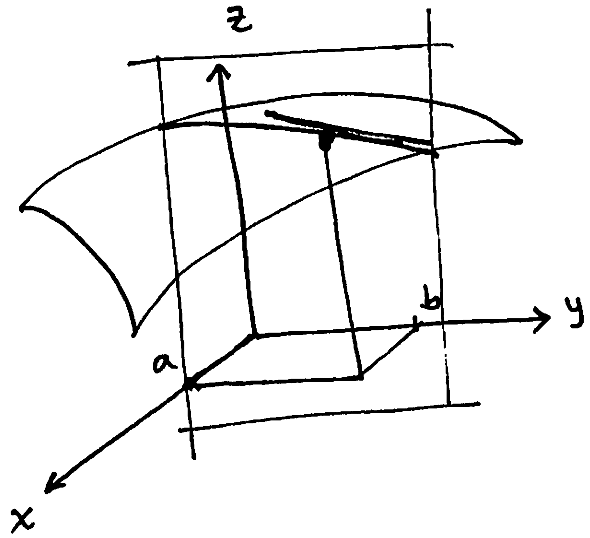
$z = f(x,y)$



partial derivative of f with respect to x at (a,b) we mean, keep y constant equal to b and compute the change in z as x change

$y=b$ plane parallel to the plane xz.

When we intersect the surface $z=f(x,y)$ with the plane $y=b$ we get a curve, and there the derivative is the slope of the tangent.



In a similar way, the partial derivative of f with respect to y at (a,b) we mean, keep x constant equal to a and compute the change in z as y changes.

- $x=a$ plane parallel to the plane yz
- when we intersect the surface $z=f(x,y)$ with the plane $x=a$ we get a curve, and there the derivative is the slope of the tangent.

Partial derivative of $z=f(x,y)$ with respect to x at (a,b)

$$\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

(x changes, y constant)

Partial derivative of $z=f(x,y)$ with respect to y at (a,b)

$$\lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

(x constant, y changes)

In general, we have:

Partial derivative of $z = f(x,y)$ with respect to x

$$\lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

Partial derivative of $z = f(x,y)$ with respect to y

$$\lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Notation: f_x, $\frac{\partial f}{\partial x}$

f_y, $\frac{\partial f}{\partial y}$

<u>Ex.</u> Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = x^3 + 5x^2y^4$

$\frac{\partial f}{\partial x}$ means, vary x, maintain y constant

$$\boxed{\frac{\partial f}{\partial x} = 3x^2 + 10xy^4}$$

$\frac{\partial f}{\partial y}$ means, vary y, maintain x constant

$$\boxed{\frac{\partial f}{\partial y} = 20x^2y^3}$$

$\frac{\partial f}{\partial x}$ is again a function that depends on x and y, so we can compute second order derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_x)_x = f_{xx} \qquad (f_x)_y = f_{xy}$$

$\frac{\partial f}{\partial y}$ depends on x and y, so, the second order derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$(f_y)_x = f_{yx} \qquad (f_y)_y = f_{yy}$$

continuing with our example:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 10xy^4\right) = 6x + 10y^4$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2 + 10xy^4\right) = 40xy^3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(20x^2y^3\right) = 60x^2y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(20x^2y^3\right) = 40xy^3$$

notice that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

mixed partial derivatives are <u>equal</u>