13.5 The Chain Rule

Thm Chain Rule (One Independent Variable)
Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Thm Chain Rule (Two Independent Variables)
Let z be a differentiable function of x and y, where x and y are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

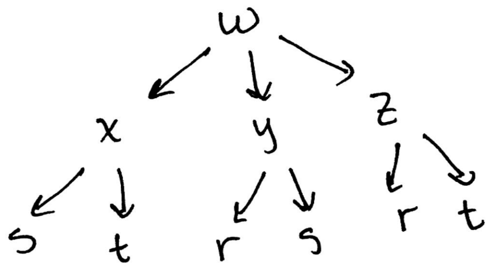
and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{tikzcd} & & z \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\ s & t & & s & t \end{tikzcd}$$

<u>Ex</u> Let $w = \sqrt{x^2+y^2+z^2}$, $x = st$, $y = rs$, $z = rt$

Find $\frac{\partial w}{\partial r}$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left[\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2y)\right](s) + $$

$$+ \left[\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2z)\right](t)$$

$$\Rightarrow \frac{\partial w}{\partial r} = \frac{ys}{\sqrt{x^2+y^2+z^2}} + \frac{zt}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(rs)\,s}{\sqrt{(st)^2+(rs)^2+(rt)^2}} + \frac{(rt)\,t}{\sqrt{(st)^2+(rs)^2+(rt)^2}}$$

$$\therefore \frac{\partial w}{\partial r} = \frac{rs^2+rt^2}{\sqrt{(st)^2+(rs)^2+(rt)^2}}$$

Implicit Differentiation.

Thm Let F be differentiable on its domain and suppose that $F(x,y)=0$ defines y as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \qquad F_y \neq 0$$

Pf. $F(x,y)=0$, lets compute $\frac{dF}{dx}$

(we are assuming that y depends on x).

$$\begin{tikzcd}
 & F \arrow[dl] \arrow[dr] & \\
x \arrow[dr] & & y \arrow[dl] \\
 & x &
\end{tikzcd}$$

Lets differentiate the equation $F(x,y)=0$ with respect to x:

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad \text{if } \frac{\partial F}{\partial y} \neq 0$$

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Ex Find $\frac{dy}{dx}$ if $x^3 + 5x^2y^3 + y^4 = 5$

Let $F(x,y) = x^3 + 5x^2y^3 + y^4 - 5$

$\therefore F(x,y)=0$, by theorem $\frac{dy}{dx} = -\frac{3x^2 + 10xy^3}{15x^2y^2 + 4y^3}$

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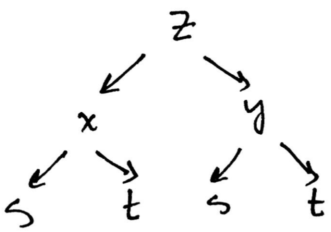
Second order partial derivatives and chain rule.

Assume $\quad z = f(x,y)$, where $x = g(s,t)$, $y = h(s,t)$

Find $\dfrac{\partial^2 z}{\partial s^2}$.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

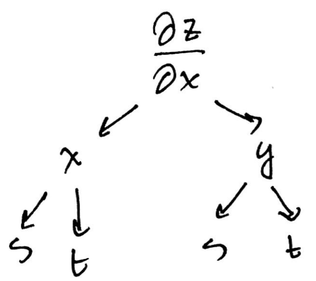
$$\Rightarrow \quad \frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial s}\left[\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right] =$$

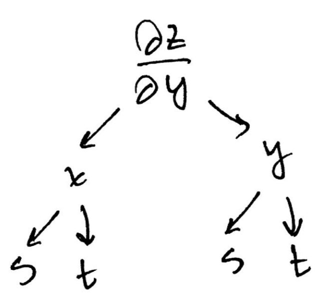
$$= \frac{\partial z}{\partial x}\cdot\frac{\partial^2 x}{\partial s^2} + \frac{\partial x}{\partial s}\cdot\underbrace{\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)}_{\text{chain rule}} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2} + \frac{\partial y}{\partial s}\cdot\underbrace{\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)}_{\text{chain rule}}$$

using product rule ~~on~~ on each term

$$= \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial x}{\partial s}\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial s}\right] +$$

$$+ \frac{\partial z}{\partial y}\cdot\frac{\partial^2 y}{\partial s^2} + \frac{\partial y}{\partial s}\left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\right]$$





$$= \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial^2 z}{\partial x^2}\cdot\left(\frac{\partial x}{\partial s}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\cdot\frac{\partial y}{\partial s}\cdot\frac{\partial x}{\partial s} +$$

$$+ \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 z}{\partial x \partial y}\cdot\frac{\partial x}{\partial s}\cdot\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial y^2}\cdot\left(\frac{\partial y}{\partial s}\right)^2$$

$$\therefore \quad \frac{\partial^2 z}{\partial s^2} = \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial^2 z}{\partial x^2}\cdot\left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 z}{\partial y \partial x}\cdot\frac{\partial x}{\partial s}\cdot\frac{\partial y}{\partial s} +$$

$$+ \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 z}{\partial y^2}\cdot\left(\frac{\partial y}{\partial s}\right)^2$$