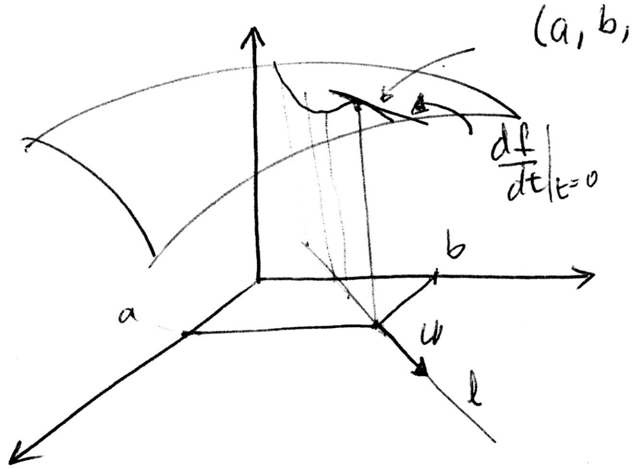
13.6 Directional Derivatives and the Gradient



$z = f(x,y)$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector

Let l be the line that passes through the point (a,b) with direction vector $\mathbf{u} = \langle u_1, u_2 \rangle$

Then l is given by:

$$\mathbf{r}(t) = \langle a, b \rangle + t\langle u_1, u_2 \rangle$$

$$\mathbf{r}(t) = \langle a + tu_1, b + tu_2 \rangle$$

notice that $\mathbf{r}'(t) = \langle u_1, u_2 \rangle$

$|\mathbf{r}'(t)| = 1$

$\therefore \mathbf{r}(t)$ is parameterized by arc length.

Then we evaluate the function at the points in the line l

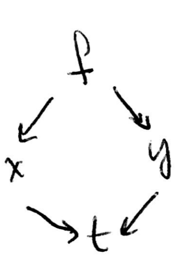
$$f(x,y) = f(a + tu_1, b + tu_2)$$

$x = a + tu_1$

$y = b + tu_2$

$t \in \mathbb{R}$

The Directional derivative of f in the direction of $\mathbf{u}$ is the rate of change of f wrt t.



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Notation: $D_{\mathbf{u}} f$

$$D_u f = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Notice that since $x = a + tu_1 \Rightarrow \frac{dx}{dt} = u_1$

$y = b + tu_2 \Rightarrow \frac{dy}{dt} = u_2$

And using the notation f_x and f_y for the partial derivatives we get:

The Directional Derivative of f at the point (a,b) in the direction of $u = \langle u_1, u_2 \rangle$ $(|u| = 1)$ is:

$$D_u f = f_x(a,b)u_1 + f_y(a,b)u_2$$

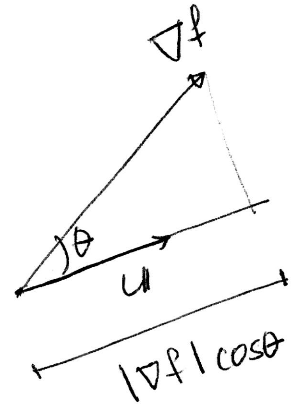
This last expression can be written as the dot product of two vectors:

$$D_u f = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$

<u>Definition</u> The gradient vector of f at (x,y) is the vector-valued function

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$\therefore \quad \boxed{D_u f(a,b) = \nabla f(a,b) \cdot u}$$

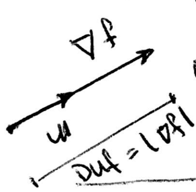


$$\nabla f \cdot u = |\nabla f| \underbrace{|u|}_{=1} \cos\theta$$

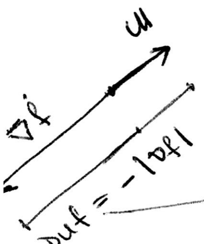
$$= |\nabla f| \cos\theta$$

- The directional derivative $D_u f$ in any given direction u is the scalar projection of the gradient vector ∇f in that direction.

Since $-1 \le \cos\theta \le 1$, then $-|\nabla f| \le |\nabla f| \cos\theta \le |\nabla f|$



∴ The maximum value that $D_u f(a,b)$ can have is when we pick u a direction that makes $\cos\theta = 1$ and $D_u f$ will be equal to $|\nabla f|$; $\cos\theta = 1$ when $\theta = 0$



Similarly, the minimum value of $D_u f(a,b)$ will be when we pick u a direction that makes $\cos\theta = -1$ and $D_u f$ will be equal to $-|\nabla f|$; $\cos\theta = -1$ when $\theta = \pi$

- The length of the gradient vector, $|\nabla f|$ is the maximum rate of increase of f.

* The vector ∇f points in the direction in which f increases most rapidly.

- The vector $-\nabla f$ points in the direction in which f decreases most rapidly.

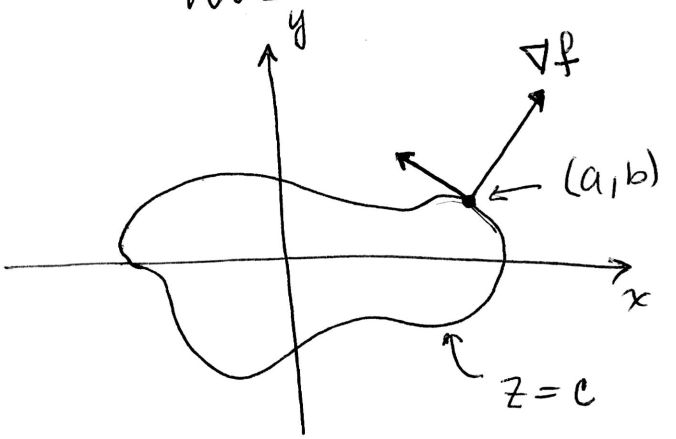
If $\theta = \pi/2$, then $D_u f = |\nabla f| \cos\theta = |\nabla f| \cos \pi/2 = 0$

- The directional derivative is zero in any direction orthogonal to ∇f.

<u>Thm</u> The gradient of f, ∇f, at a point (a,b) is orthogonal to the level curve of f that passes through (a,b).

Outline of the proof: $\nabla f = \langle f_x(a,b), f_y(a,b) \rangle$

Let $z = f(x,y)$, and let $f(a,b) = c$



level curve.

level curve has equation:

$f(x,y) = c$

$\frac{dy}{dx}$ is the slope of the tangent,

by the implicit function thm,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$\therefore$ slope of the tangent line to the level curve at (a,b) is:

$$-\frac{f_x(a,b)}{f_y(a,b)}$$

The vector: $\langle f_y(a,b), -f_x(a,b)\rangle$

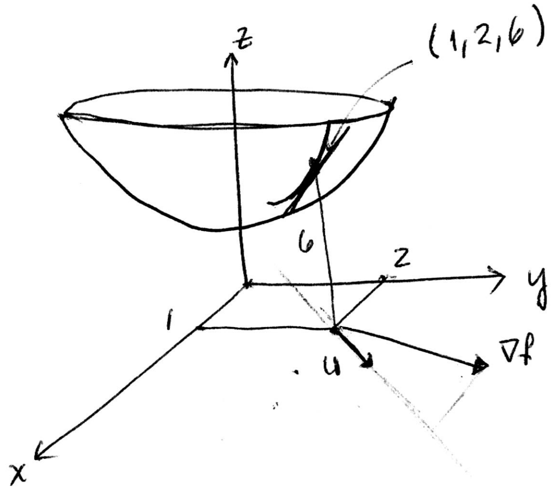
or any multiple of it, is a tangent vector to $f(x,y)=c$

Notice that $\nabla f \cdot \langle f_y(a,b), -f_x(a,b)\rangle$

$= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle f_y(a,b), -f_x(a,b)\rangle$

$= 0$

<u>Example</u> Let $z = x^2+y^2+1$, also we can write the function as $f(x,y) = x^2+y^2+1$.



Consider the point $(1,2,6)$

And let $u = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ be a given direction.

(a) Find the directional derivative of f at $(1,2)$ in the direction of $u = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$D_u f = \nabla f \cdot u$

$f_x(x,y) = 2x \Big|_{(1,2)} = 2$ $\quad\therefore \nabla f(1,2) = \langle 2, 4\rangle$

$f_y(x,y) = 2y \Big|_{(1,2)} = 4$

$\therefore D_u f = \langle 2, 4 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{6}{\sqrt{2}} \simeq 4.24$$

(b) Find the direction of maximum increase.

u must be picked in the same direction as ∇f

$$u = \frac{\nabla f}{|\nabla f|} = \frac{\langle 2, 4 \rangle}{\sqrt{20}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

(c) Find the maximum rate of increase of f at $(1,2)$.

maximum rate of increase is $|\nabla f| = \sqrt{20} = 2\sqrt{5}$

$$\simeq 4.72$$

$D_u f$ has maximum rate of increase equal to 4.72, that is at any direction u, $D_u f$ will have a value smaller than 4.72.

(d) The direction of maximum decrease is $-\frac{\nabla f}{|\nabla f|} = \langle -\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

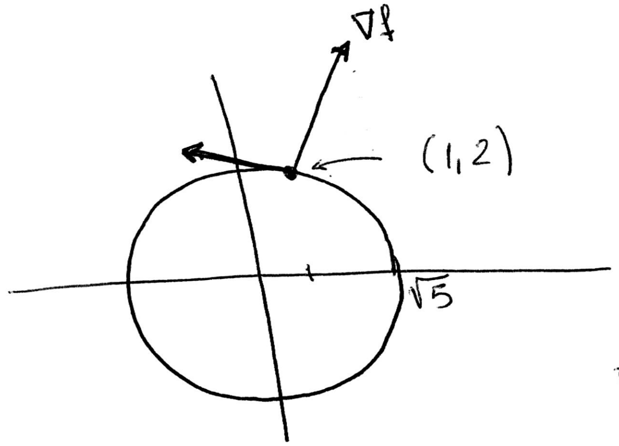
(e) The maximum rate of decrease is $-|\nabla f| = -\sqrt{20} \cong -4.72$

Now, lets consider the level curve $z=6$.

The point $(1,2)$ lies on the level curve $z=6$

$$\left.\begin{array}{l} z = x^2+y^2+1 \\ z=6 \end{array}\right\} \quad \begin{array}{l} x^2+y^2+1=6 \\ x^2+y^2=5 \end{array}$$

circle centered at the origin of radius $\sqrt{5}$

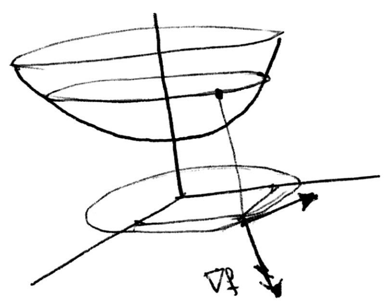


Lets compute the slope of the tangent line to $x^2+y^2=5$ at $(1,2)$.

By the implicit function thm

$$y' = -\frac{f_x(1,2)}{f_y(1,2)} = -\frac{2x|_{(1,2)}}{2y|_{(1,2)}} = -\frac{2}{4}$$

$$y' = -\frac{1}{2}$$



Lets construct a tangent vector to the curve at $(1,2)$.

Since $y' = -\frac{1}{2}$, the following vector or any multiple will be a tangent vector: $\langle -2, 1\rangle$

Notice that $\langle -2,1\rangle \cdot \nabla f|_{(1,2)} = \langle -2,1\rangle \cdot \langle 2,4\rangle = 0$

$\therefore$ $\nabla f(1,2)$ is orthogonal to the tangent vector.

We also say that ∇f is orthogonal to the level curve.