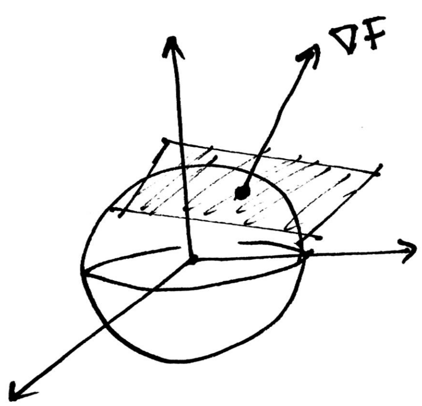
13.7 Tangent Planes

A surface in $\mathbb{R}^3$ may be defined in two different ways:

- Explicitly in the form $z = f(x,y)$ or
- Implicitly in the form $F(x,y,z) = 0$

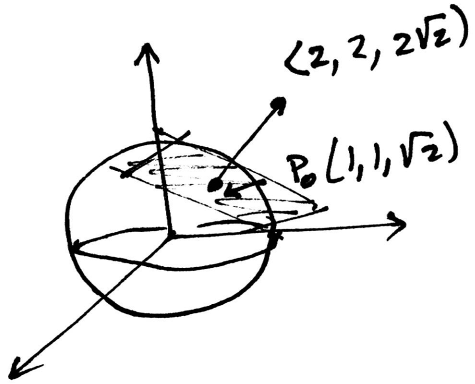
Lets first consider the surface defined implicitly.
We want to define the tangent plane to the surface at a given point, Let $P_0(a,b,c)$ be the given point.
The surface $F(x,y,z)=0$ can be considered as the level surface of the function $w = F(x,y,z)$ for $w=0$, by our discussion on gradients, we know that $\nabla F(a,b,c)$ is orthogonal to the level surface, so, we define the tangent plane to $F(x,y,z)=0$ as the plane that passes through the point $P_0(a,b,c)$ and has as normal vector the gradient $\nabla F(a,b,c)$



Tangent plane:

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

Ex Find the equation of the plane tangent to the sphere $x^2+y^2+z^2=4$ at the point $P_0(1, 1, \sqrt{2})$



Let $F(x,y,z) = x^2+y^2+z^2-4$

$$F_x = 2x \Big|_{(1,1,\sqrt{2})} = 2$$

$$F_y = 2y \Big|_{(1,1,\sqrt{2})} = 2$$

$$F_z = 2z \Big|_{(1,1,\sqrt{2})} = 2\sqrt{2}$$

$$\therefore \; 2(x-1) + 2(y-1) + 2\sqrt{2}\,(z-\sqrt{2}) = 0$$

To find the equation of the tangent plane of a surface given explicitly we do the following:

Given $z = f(x,y)$ and $P_0(a,b,c)$

we constuct: $F(x,y,z) = z - f(x,y) = 0$

Then equation of plane becomes:

$$F_x = -f_x, \quad F_y = -f_y, \quad F_z = 1$$

$$-f_x(a,b,c)(x-a) - f_y(a,b,c)(y-b) + 1\,(z-c) = 0$$

Equivalently:

$$z-c = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b).$$

Ex

Find an equation of the tangent plane to the surface $z = x^2y^2$ at the point $(2, 1, 4)$

$$z = f(x,y) = x^2 y^2$$

$$f_x = 2xy^2 \Big|_{(2,1)} = 4$$

$$f_y = 2x^2y \Big|_{(2,1)} = 8$$

$\therefore$ Equation of tangent plane

$$\underline{(z-4) = 4(x-2) + 8(y-1)}$$