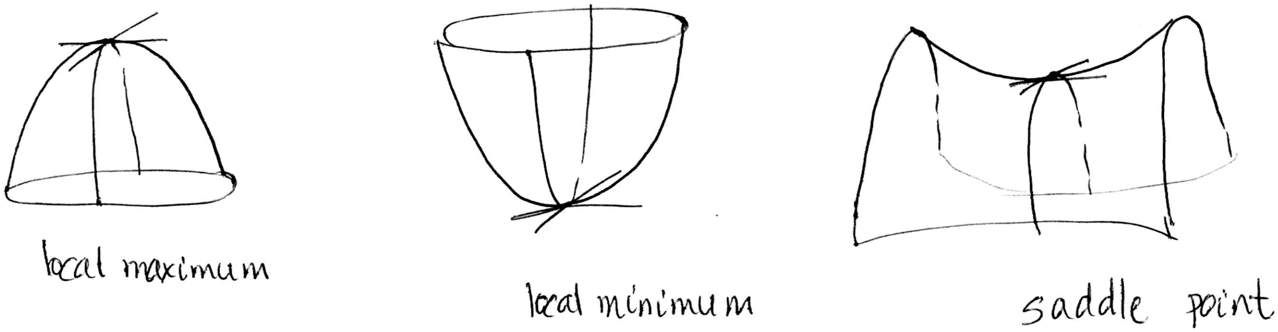
13.8 Maximum and Minimum Problems



Def. Critical Point

An interior point (a,b) in the domain of f is a critical point of f if either

1. $f_x(a,b)=0$ and $f_y(a,b)=0$ or
2. at least one of the partial derivatives f_x and f_y does not exist at (a,b).

Thm Second derivative test

Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a,b) where $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let $D = f_{xx} f_{yy} - (f_{xy})^2$ (D is called the discriminant)

1. If $D>0$ and $f_{xx}<0$ then f has a local maximum at (a,b)
2. If $D>0$ and $f_{xx}>0$ then f has a local minimum at (a,b)
3. If $D<0$ then f has a saddle point at (a,b)
4. If $D=0$ then the test is inconclusive

Ex Find and classify the critical points of

$$f(x,y) = xy + \frac{8}{y} + \frac{8}{x}$$

$f_x = y - \frac{8}{x^2}$ $\quad f_x = 0 \Rightarrow y - \frac{8}{x^2} = 0$

$f_y = x - \frac{8}{y^2}$ $\quad f_y = 0 \Rightarrow x - \frac{8}{y^2} = 0$

Notice that the domain of f is $\{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$

$\therefore$ f_x and f_y are well defined in this domain.

To find the critical points, we need to solve simultaneously:

$$y - \frac{8}{x^2} = 0 \qquad (1)$$

$$x - \frac{8}{y^2} = 0 \qquad (2)$$

From 1 $y = \frac{8}{x^2}$, plug in (2), we get

$$x - \frac{8}{\left(\frac{8}{x^2}\right)^2} = 0 \Rightarrow x - \frac{x^4}{8} = 0$$

$$\Rightarrow x\left(1 - \frac{x^3}{8}\right) = 0 \Rightarrow x = 0 \text{ or } \frac{x^3}{8} = 1$$

$\Rightarrow$ $x = 0$ or $x = 2$,

not in the domain

$\therefore$ $x = 2$ plugging in (1) $y = \frac{8}{x^2} = \frac{8}{4} = 2$

Critical point $(2,2)$

Second derivative test

$f_{xx} = \left.\frac{16}{x^3}\right|_{(2,2)} = 2$

$f_{yy} = \left.\frac{16}{y^3}\right|_{(2,2)} = 2$

$f_{xy} = 1$

$D = (2)(2) - (1)^2 = 3 > 0$

$D > 0$ and $f_{xx} = 2 > 0$

$\therefore$ The function f has a local minimum at $(2,2)$