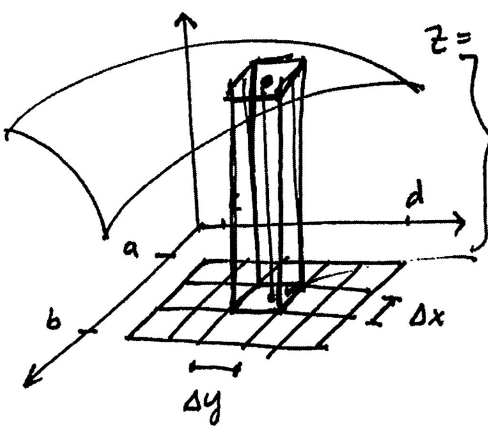
14.1 Double Integrals over Rectangular Regions.



Volume of k^{th} parallelepiped $V_k = f(x_k^*, y_k^*)\,\Delta A$ where $\Delta A = \Delta x\,\Delta y = \Delta y\,\Delta x$

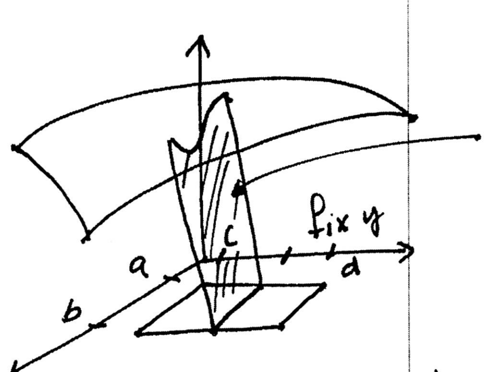
R: Rectangle on the xy-plane.

$R = \{(x,y) \mid a \le x \le b,\quad c \le y \le d\}$.

<u>Def.</u> Double integral of f over R:

$$\iint_R f(x,y)\,dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A$$

To compute the double integral we will use the Cavalieri principle, we will add cross-sectional areas.



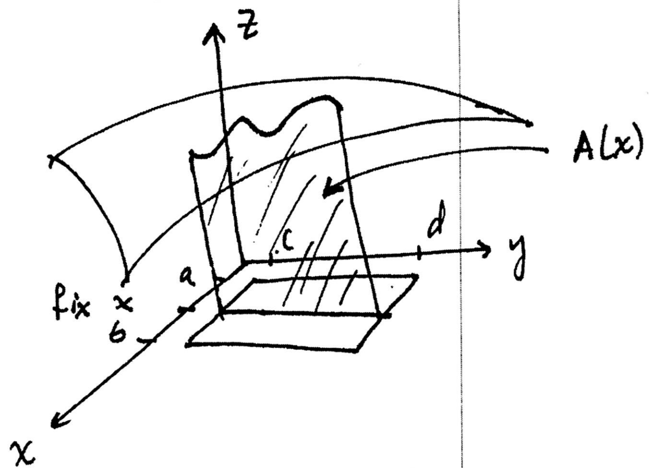
now, lets find $A(y)$:

$$A(y) = \int_a^b f(x,y)\,dx$$

$$\therefore \iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$\iint_R f(x,y)\,dA = \int_e^d A(y)\,dy$$

Similarly, we could do the following:



where $A(x)$ is:

$$A(x) = \int_c^d f(x,y)\,dy$$

$$\therefore \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$\therefore \iint_R f(x,y)\,dA = \int_a^b A(x)\,dx$$

Lets write these findings in the following theorem:

<u>Thm</u> (Fubini) Double integrals on Rectangular Region.

Let f be continuous on the rectangular region $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Ex. Evaluate the double integral

$$\iint_R (x^2 + xy)\,dA \quad \text{where } R: \{(x,y) \mid 1 \le x \le 2, \ -1 \le y \le 1\}$$

$$\iint_R (x^2+xy)\,dA = \int_{y=-1}^{1}\int_{x=1}^{2} (x^2+xy)\,dx\,dy = \int_{y=-1}^{1}\left[\left(\frac{x^3}{3} + \frac{x^2 y}{2}\right)\Bigg|_{x=1}^{2}\right]dy$$

$$= \int_{y=-1}^{1}\left[\left(\frac{8}{3} + 2y\right) - \left(\frac{1}{3} + \frac{y}{2}\right)\right]dy$$

$$= \int_{-1}^{1}\left(\frac{7}{3} + \frac{3y}{2}\right)dy = \frac{7y}{3} + \frac{3y^2}{4}\Bigg|_{-1}^{1} = \underline{\frac{14}{3}}$$

<u>Example</u> choosing a convenient order of integration

$$\iint_R x \sec^2 xy \, dA$$

$R = \{(x,y) \mid 0 \le x \le \pi/3, \ 0 \le y \le 1\}$

In this problem it is convenient to integrate wrt y first. (If we choose to integrate wrt x first, we would need to apply integration by parts in this first step ...)

$$\iint_R x \sec^2 xy \, dA = \int_0^{\pi/3} \int_0^1 x \sec^2 xy \, dy \, dx$$

$$= \int_0^{\pi/3} \left[\int_0^1 x \sec^2 xy \, dy \right] dx = \int_0^{\pi/3} \left[\int_0^x \sec^2 u \, du \right] dx$$

Let $u = xy$ $\quad y=0 \quad u=0$

$du = x \, dy \quad y=1 \quad u=x$

$$= \int_0^{\pi/3} \left[\tan u \Big|_0^x \right] dx = \int_0^{\pi/3} [\tan x - \underbrace{\tan 0}_{=0}] \, dx$$

$$= \int_0^{\pi/3} \tan x \, dx = \ln(\sec x) \Big|_0^{\pi/3} = \ln\left(\sec \tfrac{\pi}{3}\right) - \ln(\sec 0)$$

$$= \ln(2) - \ln(1) = \underline{\ln 2}$$

Average Value.

Def. The average value of an integrable function f over a region R is:

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x,y)\, dA$$

Ex. Find the average value of $f(x,y) = 4 - x - y$ over the region $R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$

area of $R = 4$

$$\therefore \bar{f} = \frac{1}{4} \int_0^2 \int_0^2 4 - x - y \; dx\, dy$$

$$= \frac{1}{4} \int_0^2 \left(4x - \frac{x^2}{2} - yx\right)\Big|_0^2 dy = \frac{1}{4} \int_0^2 (6 - 2y)\, dy$$

$$= \frac{1}{4}\left[6y - y^2 \Big|_0^2\right] = \frac{1}{4}[8] = \underline{2}$$