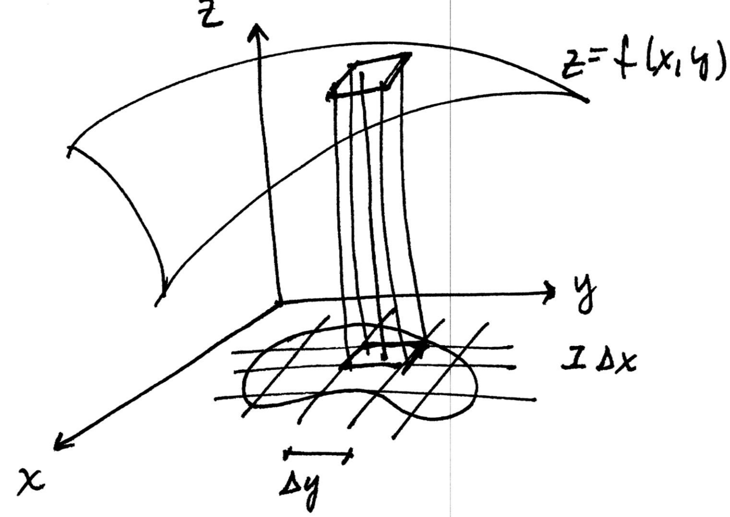
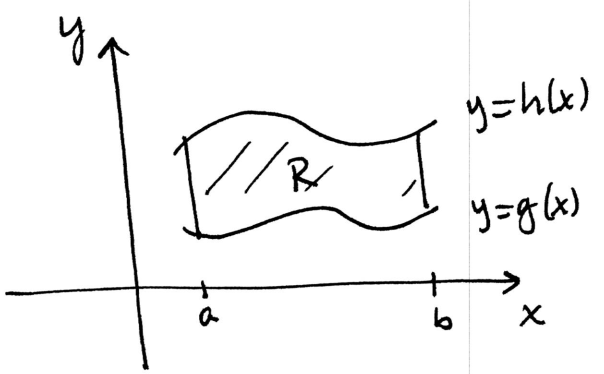
14.2 Double Integrals over General Regions



$$\iint_R f(x,y)\, dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A$$

Most of the double integrals that we will encounter fall into two categories.

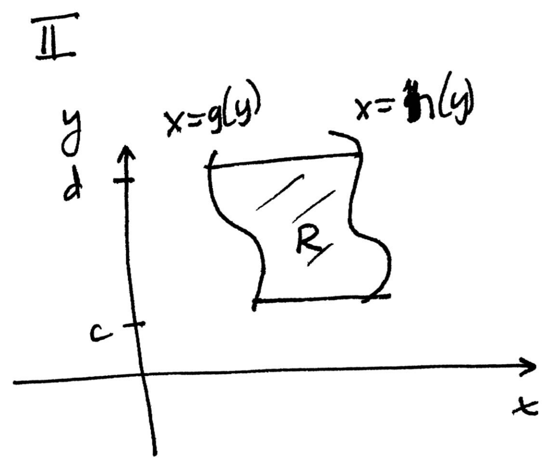
I.



Region bounded by two functions of x, where x varies from a to b

$$R = \{ (x,y) \mid g(x) \le y \le h(x),\ a \le x \le b \}$$

II

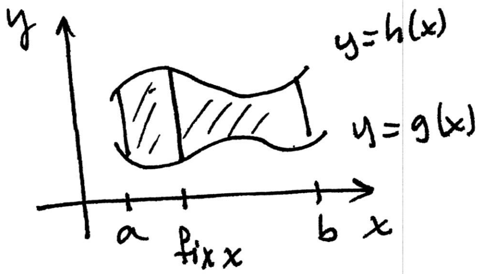


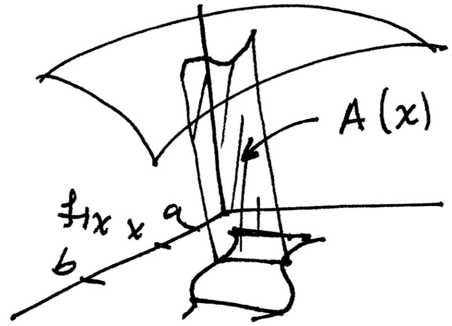
Region bounded by two functions of y, where y varies from c to d

$$R = \{ (x,y) \mid g(y) \le x \le h(y),\ c \le y \le d \}.$$

To evaluate double integrals over these regions, we will use the same Cavalieri principle, or general slicing method...

Region type I



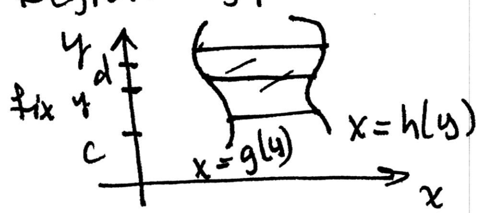


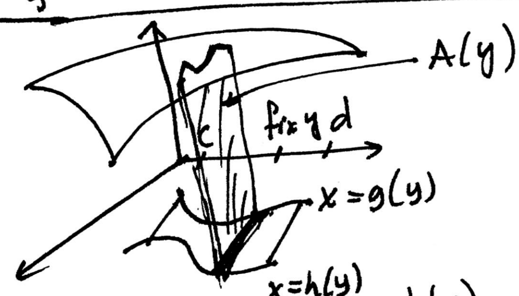
$$\iint_R f(x,y)\,dA = \int_a^b A(x)\,dx$$ where $A(x)$ can be computed as

$$A(x) = \int_{g(x)}^{h(x)} f(x,y)\,dy$$

$$\therefore \iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Region type II





$$\iint_R f(x,y)\,dA = \int_c^d A(y)\,dy$$ where $A(y) = \int_{g(y)}^{h(y)} f(x,y)\,dx$

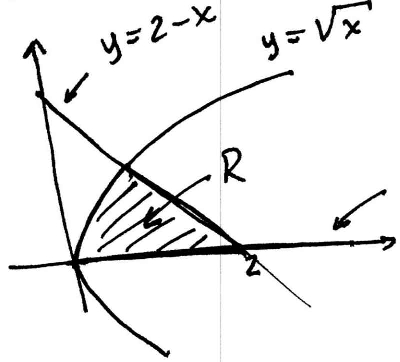
$$\therefore \iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Ex. Evaluate the following integral.

$$\iint_R 12y \, dA \qquad R \text{ is bounded by } y = 2 - x, \quad y = \sqrt{x}, \text{ and } y = 0$$

- first we will draw the region, this will be an important aid for finding the limits of integration correctly.

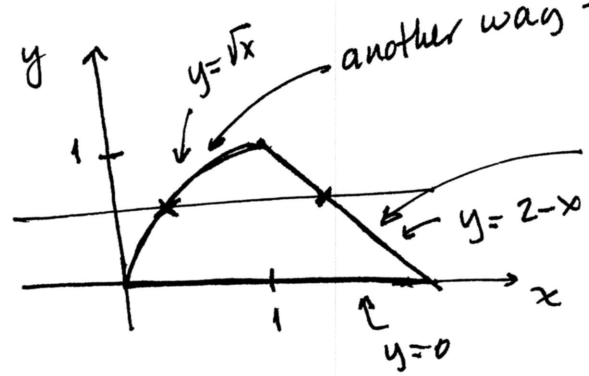
R:



- We will integrate first with respect to x, and then with respect to y.

if we integrate first with respect to x, we draw a line parallel to the x-axis and find where this line enters the region (this will be our lower limit of integration) and then find where the line leaves the region (this will be the upper limit of integration)
Then we will hold this same line and move it in the direction of y now until we sweep the whole region and those will be the limits of integration of the last integral...

Example continues...



Since we are going to integrate wrt x first we need to write the limits of integration as x = function of y ...

$$\iint_R 12y \, dA = \int \int_{x=y^2}^{2-y} 12y \, dx \, dy$$

To find the last limits of integration, we need to find where $y=\sqrt{x}$ and $y = 2-x$ intersect, and we see that they do when $x=1$ and $y=1$. Therefore to sweep the whole region, y must vary from 0 to 1.

$$\therefore \iint_R 12y \, dA = \int_{y=0}^{1} \int_{x=y^2}^{2-y} 12y \, dx \, dy$$

$$\therefore \iint_R 12y\, dA = \int_{y=0}^{1} \int_{x=y^2}^{2-y} 12y\, dx\, dy = \int_0^1 \left[12yx \Big|_{x=y^2}^{2-y} \right] dy$$

$$= \int_0^1 12y\left[(2-y) - y^2\right] dy = 12\int_0^1 2y - y^2 - y^3\, dy = 12\left[y^2 - \frac{y^3}{3} - \frac{y^4}{4} \right]\Big|_0^1$$

$$= \underline{5}$$

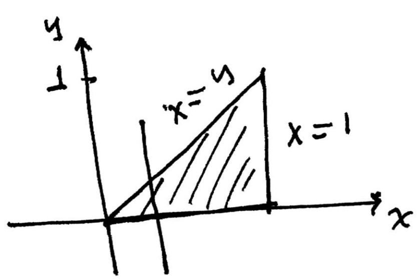
Ex Change the order of integration of the following:

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

Lets draw the region of integration first.

$x = y$ and then y varies from 0 to 1.

$x = 1$



Now lets write the given integral but integrating first wrt y and then wrt x.

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

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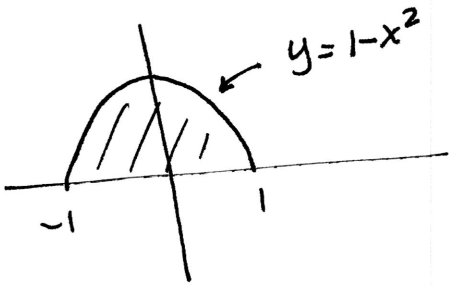
Areas of Regions by double integrals.

$$\text{area of } R = \iint_R 1 \, dA$$

Find the area of the region bounded by $y = 1 - x^2$ on $[-1, 1]$.



<u>Sol 1</u>. Area $= \int_{-1}^{1} (1-x^2)\, dx$

<u>Sol 2</u> Area $= \int_{-1}^{1} \int_{0}^{1-x^2} dy\, dx$