14.3 Double Integrals in Polar Coordinates.

It is often more convenient to describe the boundaries of a region by using polar coordinates r, θ than by using rectangular coordinates x, y.

... How to express $\iint_R f(x,y)\,dA$ in polar coordinates.

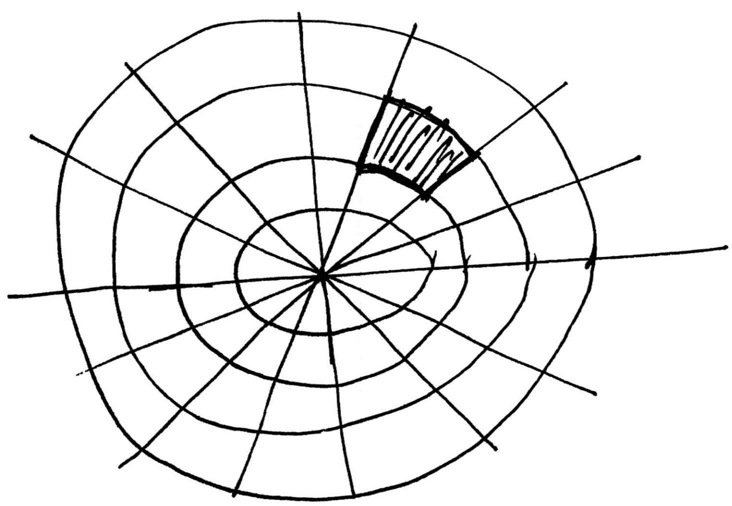
The integrand is easy to transform by using the equations $x = r\cos\theta$, $y = r\sin\theta$ to write $f(x,y)$ as a function of r and θ,

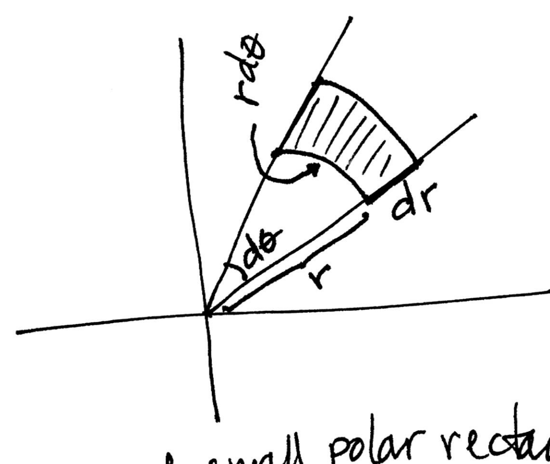
$$f(x,y) = f(r\cos\theta, r\sin\theta)$$

But what do we do with dA?

dA meant the area of a small rectangle $dx\,dy$

so, how do we compute the area of a small polar rectangle ...





area of small polar rectangle
$= (r\,d\theta)\,dr$

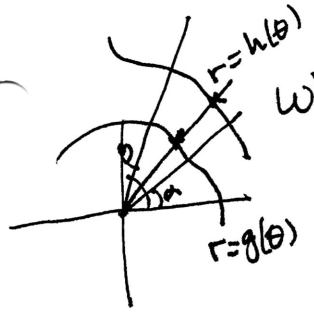
$$\therefore \iint_R f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Many of the regions R we deal with are radially simple, in the sense that they can be described by inequalities of the form

$$\alpha \le \theta \le \beta, \qquad g(\theta) \le r \le h(\theta)$$

<u>Thm</u> Let f be continuous on the region in the xy-plane:

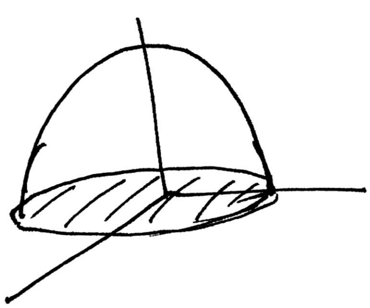
$$R = \{(r,\theta) \mid 0 \le g(\theta) \le r \le h(\theta), \ \alpha \le \theta \le \beta\}$$



where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

<u>Ex.</u> Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane



Intersection of $z = 9 - x^2 - y^2$ and the xy-plane $z = 0$ is $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9$

$\therefore$ The region of integration is the disk:

$$R: \{(x,y) \mid x^2 + y^2 \le 9\}$$

In polar coordinates R is:

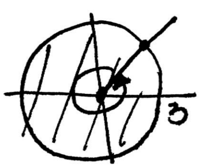
$$R = \{ (r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi \}$$

$$\therefore\ V = \int_0^{2\pi} \int_0^{3} (9 - r^2)\, r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^3 \right] d\theta$$

$$= \int_0^{2\pi} \left(\frac{81}{2} - \frac{81}{4} \right) d\theta = \frac{81}{4}(2\pi) = \underline{\underline{\frac{81}{2}\pi}}$$

$f(x,y) = 9 - x^2 - y^2 = 9 - r^2$

$dA = r\,dr\,d\theta$



$0 \le r \le 3$

$0 \le \theta \le 2\pi$

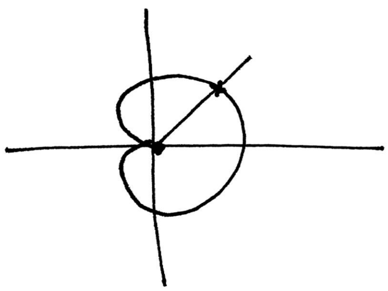
Area in polar regions.

The area of the polar region $R = \{ (r,\theta) \mid 0 \le g(\theta) \le r \le h(\theta),$
$\alpha \le \theta \le \beta \}$

where $0 < \beta - \alpha \le 2\pi$ is

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r\,dr\,d\theta$$

Ex Find the area of the region R enclosed by the cardioid $r = 1+\cos\theta$



$$A = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^2}{2}\Big|_0^{1+\cos\theta}\right)d\theta = \int_0^{2\pi}\frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1+2\cos\theta+\cos^2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left[(2\pi + 0 + \pi + 0) - (0)\right]$$

$$= \underline{\frac{3\pi}{2}}$$

Ex Compute $\displaystyle\int_0^\infty e^{-x^2}\,dx$:)

Let $\displaystyle I = \int_0^\infty e^{-x^2}\,dx$

Since it doesn't matter what letter we use for the variable of integration, we have:

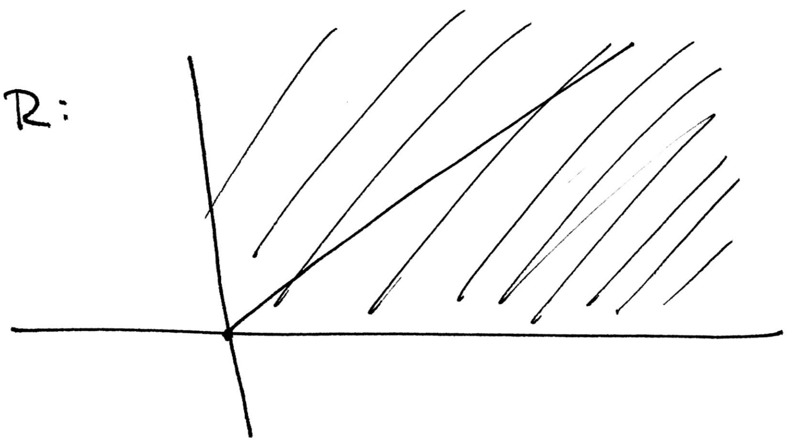
$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

This can be written in the form:

$$I^2 = \int_0^\infty \left(\int_0^\infty e^{-x^2} dx \right) e^{-y^2} dy = \int_0^\infty \left(\int_0^\infty e^{-x^2} e^{-y^2} dx \right) dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

R:



$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \Big|_0^\infty \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(0 - \left(-\frac{1}{2}\right)\right) d\theta = \frac{1}{2} \theta \Big|_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore \quad I^2 = \pi/4 \quad \Rightarrow \quad I = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

This formula is especially remarkable because it is known that the indefinite integral $\int e^{-x^2} dx$ is impossible to express as an elementary function...