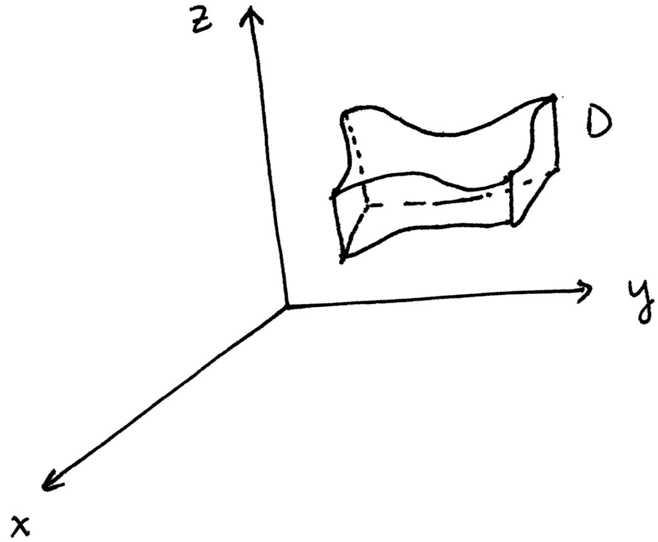
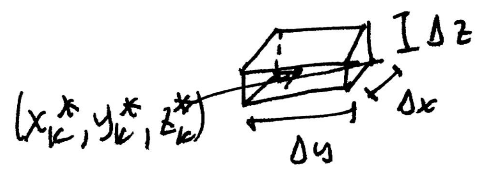
14.4 Triple Integrals.



Let D be a region in $\mathbb{R}^3$ and f a function of three variables:

$$w = f(x,y,z)$$

Subdivide the region D in small boxes



The volume of this box is ΔV, notice that now we have 6 options to compute ΔV.

$$\begin{aligned} \Delta V &= \Delta x \Delta y \Delta z \\ &= \Delta x \Delta z \Delta y \\ &= \Delta y \Delta x \Delta z \\ &= \Delta y \Delta z \Delta x \\ &= \Delta z \Delta x \Delta y \\ &= \Delta z \Delta y \Delta x \end{aligned}$$

Pick a point in this box, say (x_k^*, y_k^*, z_k^*), then evaluate the function: $f(x_k^*, y_k^*, z_k^*)$,

now, we form the product:

$$f(x_k^*, y_k^*, z_k^*)\,\Delta V$$

Finally, we form the sum of these products over all the boxes that lie inside D

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta V,$$

at the limit we obtain what we call the triple integral.

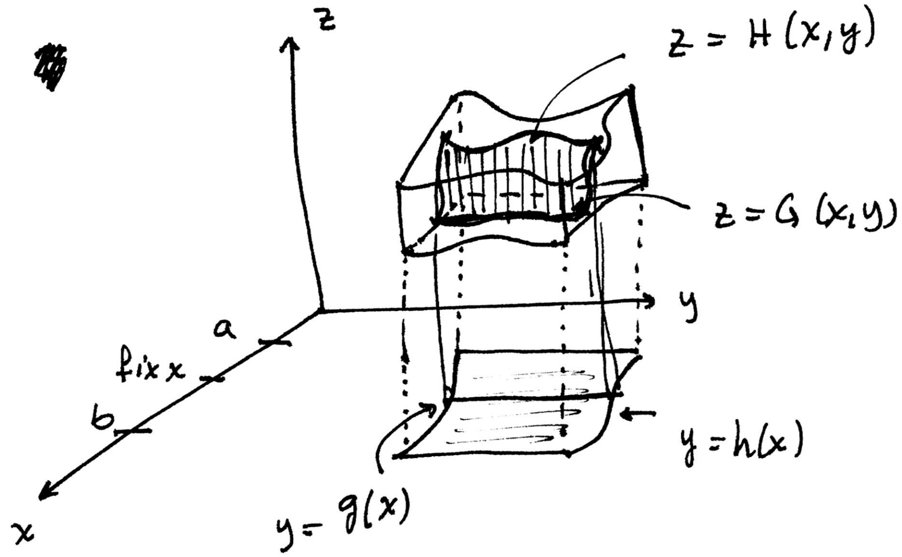
<u>**Def**</u> Triple integral of f over D:

$$\iiint_D f(x,y,z)\,dV = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta V$$

Finding limits of integration.

Let D in $\mathbb{R}^3$ be a region defined as follows:

$$D = \{ (x,y,z) \mid a \le x \le b, \quad g(x) \le y \le h(x), \quad G(x,y) \le z \le H(x,y) \}$$



fix x, then by the general slicing principle

$$\iiint_D f(x,y,z)\, dV = \int_a^b V(x)\, dx$$

now, lets compute $V(x)$:

$V(x)$ is the volume of the solid under the function $f(x,y,z)$ (x-fix) over the 2-dimensional region defined by y and z.

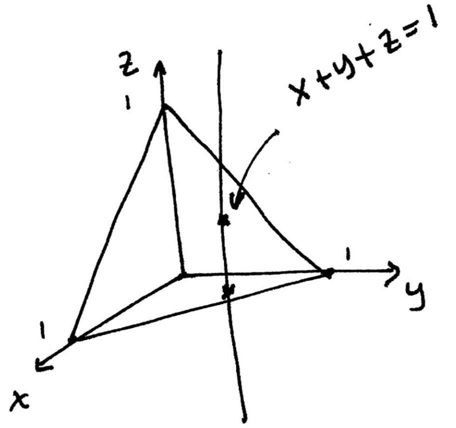
$$V(x) = \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz\, dy$$

$$\therefore \iiint_D f(x,y,z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

Other orders of integration are often possible, and the order we choose will depend on the specific problem we want to solve.

<u>Ex.</u> Compute $\iiint_D z \, dV$

where D is the region bounded by the coordinate planes and the plane $x+y+z=1$



Lets integrate first with respect to ~~y~~ z, then we trace a line parallel to the z-axis and see where the line enters the region (this will be the lower limit of integration) and then see where the line leaves the region (this will be the upper limit of integration).

So far, we have:

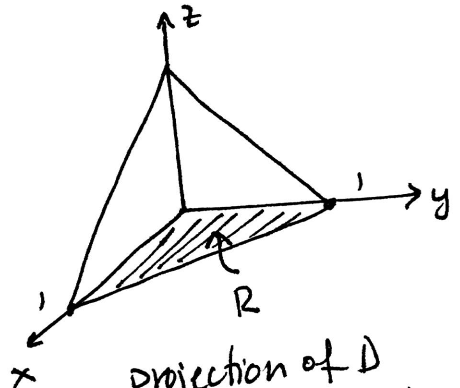
$$\iiint_D z \, dV = \underbrace{\iint}_{R} \int_0^{1-x-y} z \, dz \, \underline{\underline{dA}}$$

$x+y+z=1$

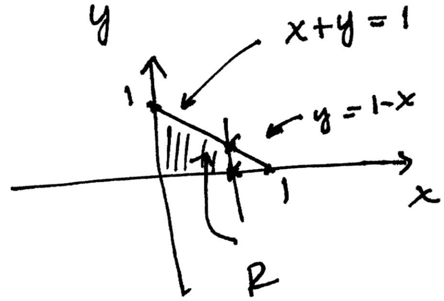
$\Rightarrow z = 1-x-y$

To compute the other limits of integration, we project the region D into the xy-plane.

Example continues...



projection of D
into the xy-plane.



$$\therefore \iiint_D z \, dV = \int_0^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2} \Big|_0^{1-x-y} dy \, dx = \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2} \, dy \, dx =$$

$$\int \frac{(1-x-y)^2}{2} dy = -\int \frac{u^2}{2} du = -\frac{u^3}{6} = -\frac{(1-x-y)^3}{6}$$

Let $u = 1-x-y$
$du = -dy$
$-du = dy$

$$= \int_0^1 -\frac{(1-x-y)^3}{6} \Big|_{y=0}^{1-x} dx = -\frac{1}{6} \int_0^1 \left[0 - (1-x)^3 \right] dx$$

$$= \frac{1}{6} \int_0^1 (1-x)^3 \, dx = \frac{1}{6} \left[-\frac{(1-x)^4}{4} \Big|_0^1 \right] = -\frac{1}{6} \left[\frac{(1-x)^4}{4} \Big|_0^1 \right]$$

$$= -\frac{1}{6} \left(0 - \frac{1}{4} \right) = \underline{\frac{1}{24}}$$

Volumes of Regions by triple integrals.

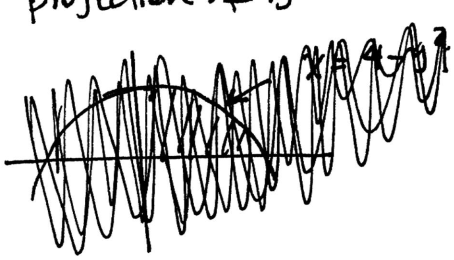
$$\text{Volume of } D = \iiint_D 1 \, dV$$

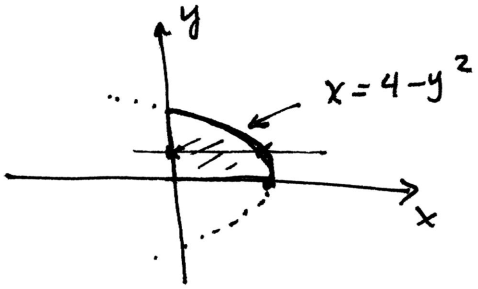
<u>Ex.</u> Find the volume of the region in the first octant bounded by the cylinder $x = 4-y^2$, and the planes $y = z$, $x = 0$, $z = 0$.

$$V = \iiint_D 1 \, dV = \int_0^2 \int_0^{4-y^2} \int_0^y dz \, dx \, dy$$

$$D: \{ (x,y,z) \mid 0 \le z \le y, \ 0 \le x \cancel{y} \le 4-y^2, \quad 0 \le y \le 2 \}$$

projection of D:





$$\int_0^2 \int_0^{4-y^2} z \Big|_0^y \, dx dy = \int_0^2 \int_0^{4-y^2} y \, dx \, dy$$

$$= \int_0^2 yx \Big|_0^{4-y^2} dy = \int_0^2 y(4-y^2) \, dy$$

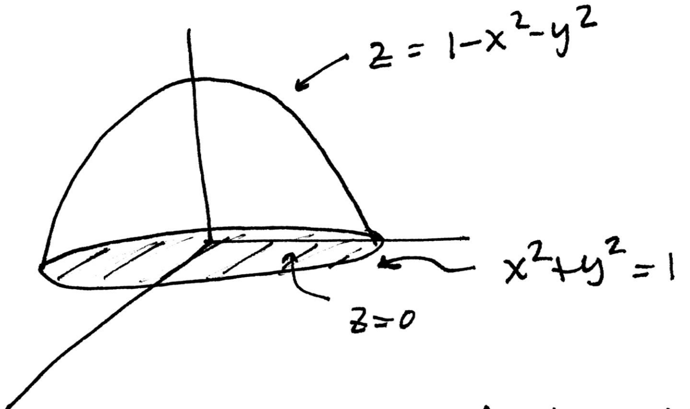
$$= \int_0^2 4y - y^3 \, dy = 2y^2 - \frac{y^4}{4} \Big|_0^2 = 8 - 4 = \underline{4}$$

Ex. Change the order of integration:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{1-x^2-y^2} f(x,y,z)\, dz\, dy\, dx$$

$$= \int \int \int f(x,y,z)\, dx\, dy\, dz$$

$z = 1-x^2-y^2$ is a paraboloid ...



project unto xy-plane.

$z=0 \Rightarrow 0 = 1-x^2-y^2$

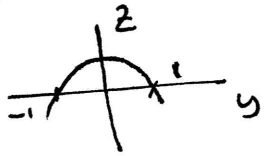
$\Rightarrow x^2+y^2 = 1$

To integrate wrt x first, solve for x from $z = 1-x^2-y^2$

$\Rightarrow x = \pm\sqrt{1-z-y^2}$

Then lets project $z = 1-x^2-y^2$ unto the yz-plane, set $x=0$

$z = 1-y^2$



$\Rightarrow y^2 = 1-z$

$\Rightarrow y = \pm\sqrt{1-z}$

$\therefore$ the limits of integration with this order will be:

$$\int_{0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{-\sqrt{1-z-y^2}}^{\sqrt{1-z-y^2}} f(x,y,z)\, dx\, dy\, dz$$