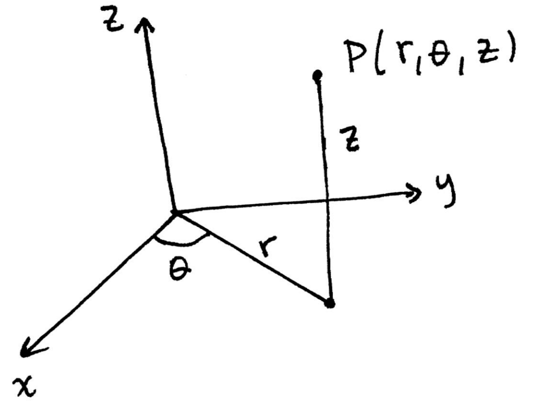
14.5 Triple integrals in cylindrical and spherical coordinates.

Cylindrical coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

cylindrical to cartesian

$x^2 + y^2 = r^2$

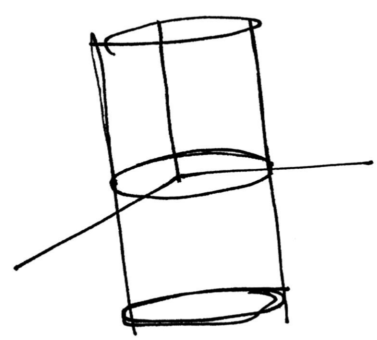
$\tan\theta = \dfrac{y}{x}$

$z = z$

Cartesian to cylindrical

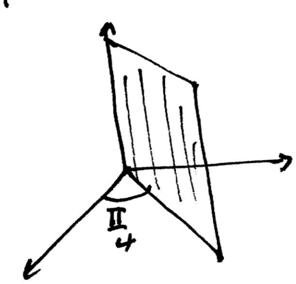
The equation $r = 2$ is a cylinder, we interpret it as

r is 2, for all θ, and all z:



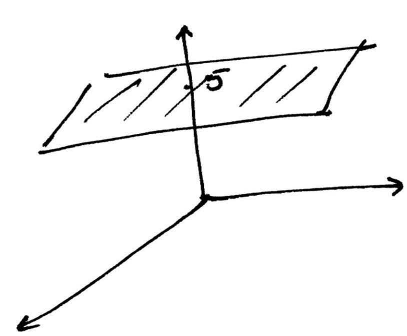
$\theta = \frac{\pi}{4}$ is a plane, we interpret this equation as

θ is $\frac{\pi}{4}$ for all r and all z:



$z = 5$ is a plane, we interpret this equation as

z is 5 for all r and all θ

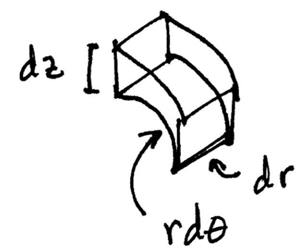


A triple integral would transform as follows:

$$\iiint_D f(x,y,z)\, dV = \iiint_D f(r,\theta,z)\; r\,dz\,dr\,d\theta$$

The element of volume in cylindrical coordinates is

$$dV = (r\,d\theta)(dr)(dz)$$
$$= r\,dz\,dr\,d\theta.$$

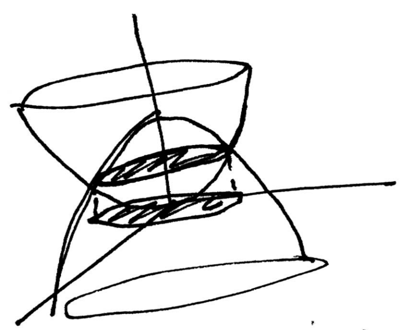


<u>Ex</u> Find the volume between the paraboloids:

$$z = x^2+y^2 \quad \text{and} \quad z = 8 - x^2 - y^2$$

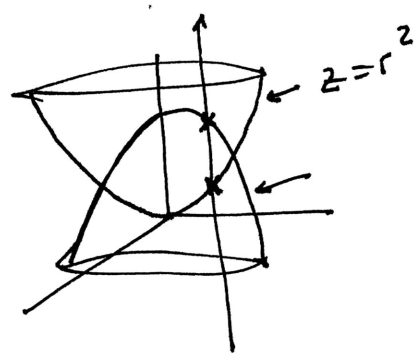
Using cylindrical coordinates we get:

$$z = r^2, \qquad z = 8 - r^2$$



So far our triple integral would be:

$$V = \iiint_{r^2}^{8-r^2} 1\; \underbrace{r\,dz\,dr\,d\theta}_{dV}$$

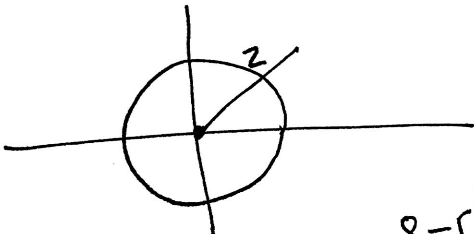


To find the other limits of integration, we intersect the two paraboloids:

$z = r^2, \quad z = 8 - r^2$

intersection: $r^2 = 8 - r^2 \Rightarrow 2r^2 = 8 \Rightarrow r^2 = 4 \Rightarrow r = 2$

$\therefore$ they intersect on a circle of radius 2, we draw this region in the xy-plane:



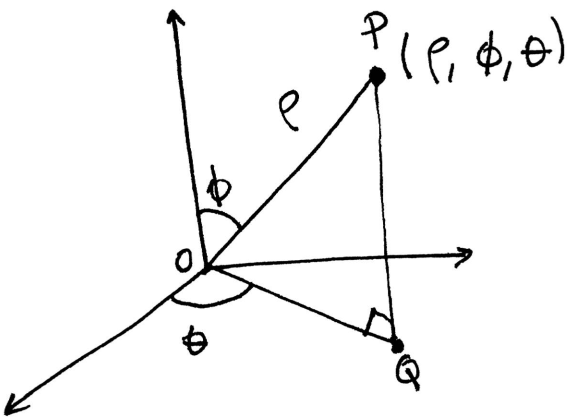
so r varies from 0 to 2
and θ varies from 0 to 2π

$$\therefore V = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 rz\Big|_{r^2}^{8-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(8-r^2-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (8r-2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} 4r^2 - \frac{r^4}{2}\Big|_0^2 d\theta = \int_0^{2\pi}(16-8)\,d\theta = 8\theta\Big|_0^{2\pi} = \underline{16\pi}$$

Spherical coordinates



$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

ρ: distance from origin to P

ϕ: angle between positive z-axis and the line OP

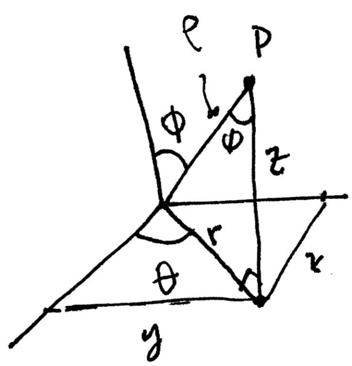
θ: same angle as polar coordinates, project the point P to the xy-plane, θ is the angle from the positive x-axis to the line OQ

$x = r\cos\theta$

$y = r\sin\theta$

$\sin\phi = \dfrac{r}{\rho}$

$\cos\phi = \dfrac{z}{\rho}$



$\therefore$ $x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

spherical coordinates to cartesian coordinates

$x^2 + y^2 + z^2 = \rho^2$

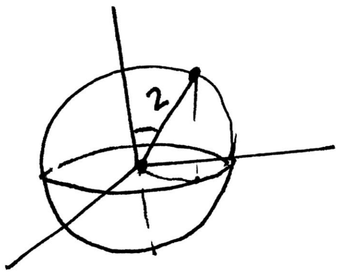
Use trigonometry to find ϕ and θ

cartesian to spherical

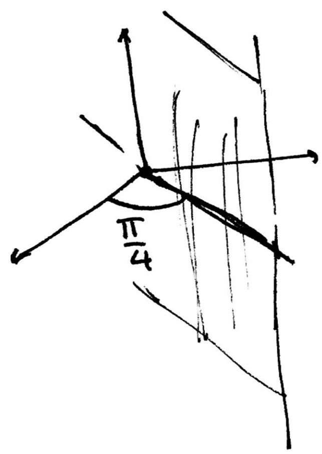
$\rho = 2$ represents a sphere of radius 2,

we interpret this equation as:

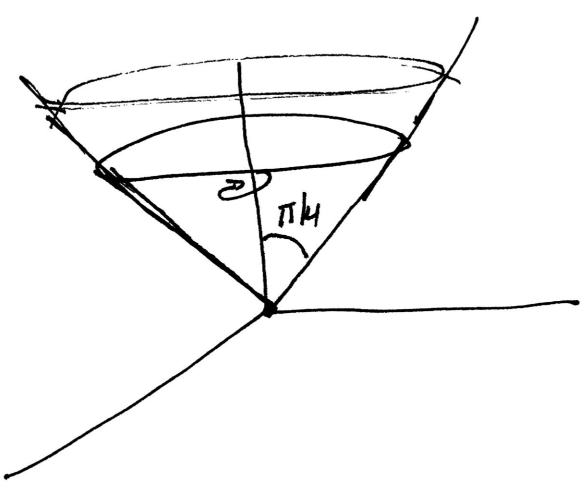
ρ is 2 for all values of θ and all values of ϕ

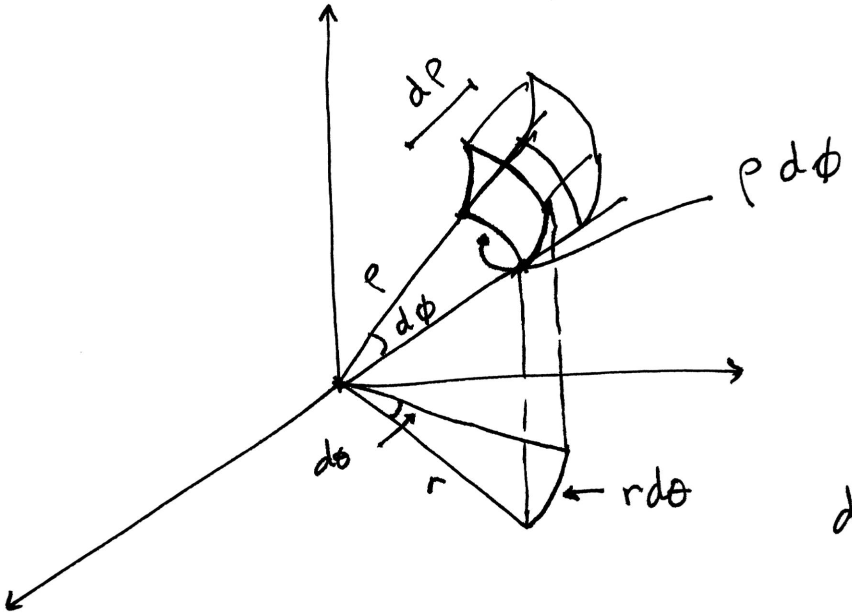


$\theta = \frac{\pi}{4}$ represents a plane, θ is $\frac{\pi}{4}$, $\forall \rho$ and $\forall \phi$



$\phi = \frac{\pi}{4}$ represents a cone, ϕ is $\frac{\pi}{4}$, $\forall \rho$ and $\forall \theta$





Element of volume in spherical coordinates

$dV = (\rho\, d\phi)(r\, d\theta)(d\rho)$

using that $r = \rho \sin\phi$

we obtain:

$dV = (\rho\, d\phi)(\rho \sin\phi\, d\theta)(d\rho)$

$= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

A triple integral in spherical coordinates:

$$\iiint_D f(x,y,z)\, dV = \iiint_D f(\rho, \phi, \theta) \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

Ex A wedge is cut from a solid sphere of radius a by two planes that intersect on a diameter. If α is the angle between the planes, find the volume of the wedge.

$$V = \int_0^{\alpha} \int_0^{\pi} \int_0^{a} 1 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\alpha} \int_0^{\pi} \frac{\rho^3}{3} \sin\phi \Big|_0^a d\phi\, d\theta$$

$$= \int_0^{\alpha} \int_0^{\pi} \frac{a^3}{3} \sin\phi\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{\alpha} (-\cos\phi)\Big|_0^{\pi} d\theta$$

$$= \frac{a^3}{3} \int_0^{\alpha} (1+1)\, d\theta = \frac{2a^3}{3}\, \theta \Big|_0^{\alpha} = \frac{2a^3\alpha}{3}$$

Ex Find the volume bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=16$

In spherical coordinates

$$x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$

$$= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

$$= \rho^2\sin^2\phi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{=1}) = \rho^2\sin^2\phi$$

$\therefore$ $z = \sqrt{x^2+y^2}$ is:

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$

$\rho\cos\phi = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi \Rightarrow 1 = \tan\phi$

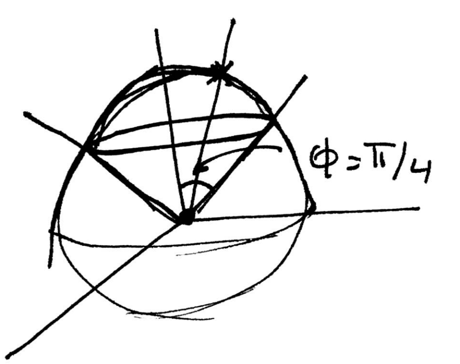
$\Rightarrow \phi = \frac{\pi}{4}$.

Cone $z=\sqrt{x^2+y^2}$ in spherical is $\phi = \frac{\pi}{4}$.

the sphere $x^2+y^2+z^2=16$ in spherical is $\rho = 4$

so ~~then~~ we have:

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4} 1\cdot \rho^2\sin\phi \, d\rho\, d\phi\, d\theta = \ldots\ldots = \frac{128\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$



ρ varies from 0 to 4.

ϕ varies from 0 to $\frac{\pi}{4}$

and θ from 0 to 2π

Ex Volume of a sphere of radius a:

$x^2 + y^2 + z^2 = a^2$

In spherical coordinates: $\rho = a$

$$\therefore V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3}\Big|_0^a \sin\phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} d\theta = \frac{a^3}{3} \int_0^{2\pi} (1+1) \, d\theta$$

$$= \frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{2a^3}{3}(2\pi) = \frac{4}{3}\pi a^3$$