16.7 Change of Variables

We used change of variables in calc I, to simplify a single variable integral.

Ex: $\displaystyle\int_{x=0}^{x=1} 2\sqrt{2x+1}\,dx$

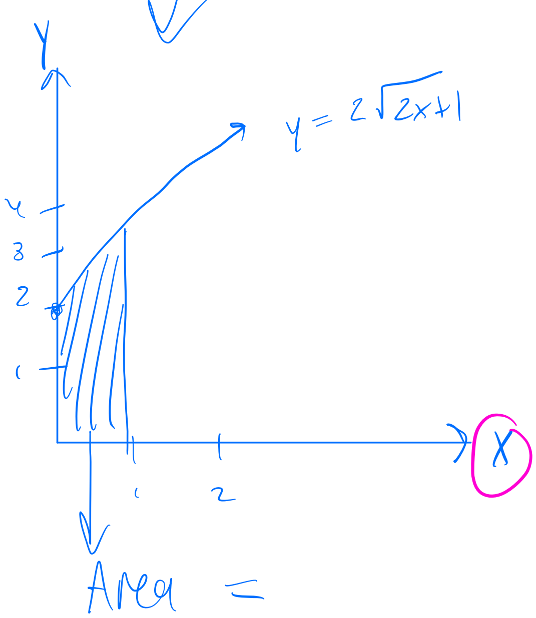
$u = 2x+1$

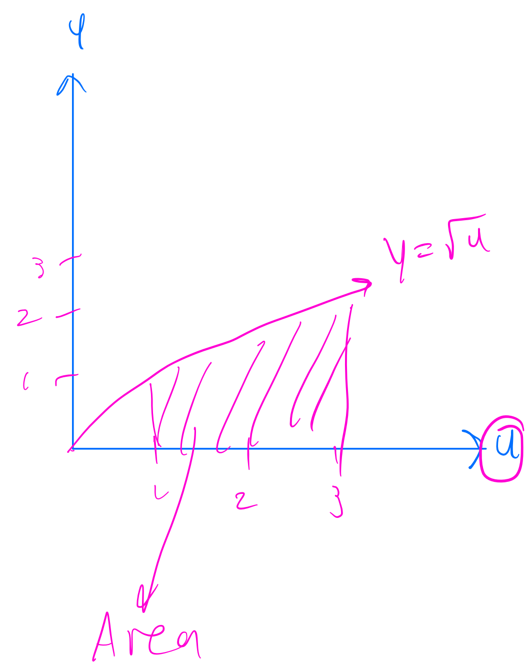
$du = 2x\,dx$

$$\int_{x=0}^{x=1} 2\sqrt{2x+1}\,dx = \int_{u=1}^{u=3} \sqrt{u}\,du$$

x=1 into 2x+1 = u

x=0 into 2x+1 = u

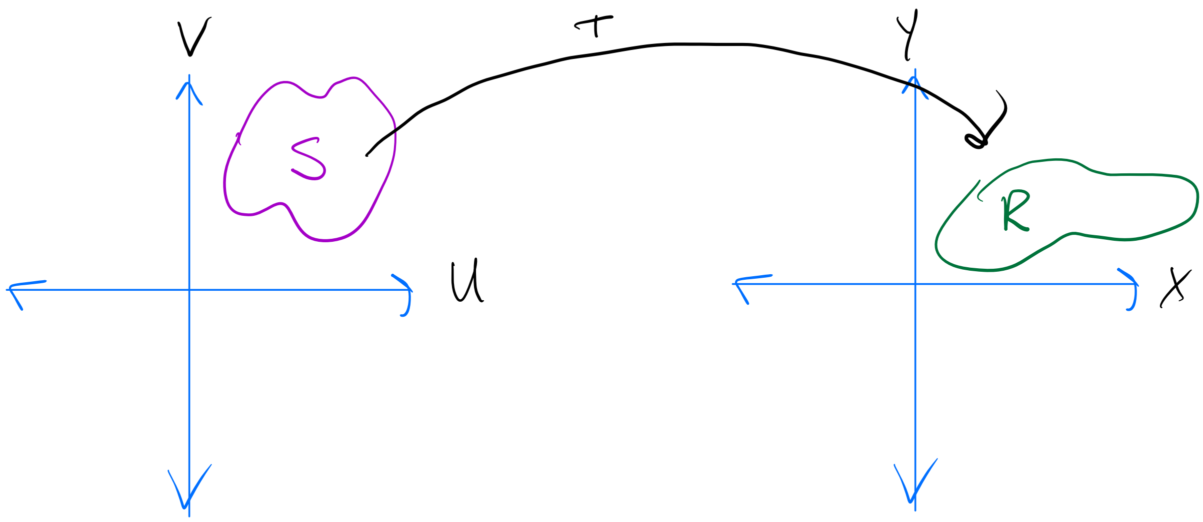




Similarly, double & triple integrals can be simplified through a change of variables (switching to polar coordinates is an example of this.)

A Transformation of 2 variables is written compactly as $(x,y) = T(u,v)$

$$T: x = g(u,v) \quad y = h(u,v)$$

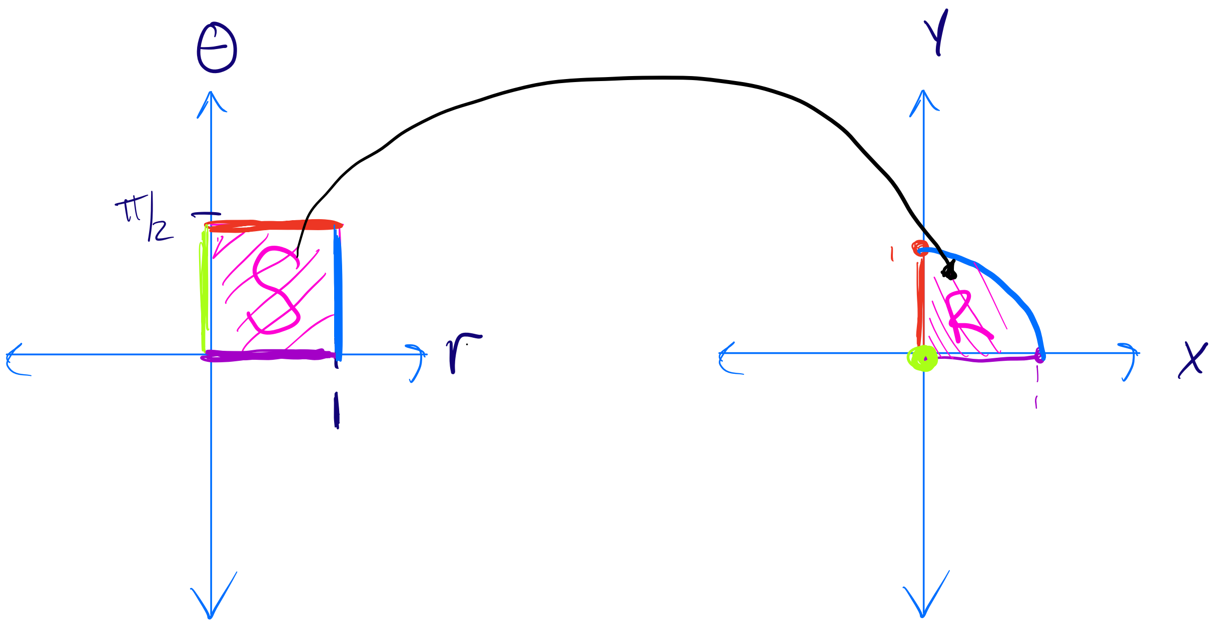


Ex: $T: \; x = g(r,\theta) = r\cos\theta$

$$y = h(r,\theta) = r\sin\theta$$

Find the image of this transformation of the rectangle

$$S = \{(r,\theta) : 0 \le r \le 1, \; 0 \le \theta \le \pi/2\}$$



Fix $\theta = 0$ go from $r=0$ to $r=1$

$$x = r\cos(0) = r$$
$$y = r\sin(0) = 0$$

$x = r$
$y = 0$

Fix $\theta = \pi/2$ go from $r=0$ to $r=1$

$x = r\cos(\pi/2) = 0$
$y = r\sin(\pi/2) = r$ } $x=0$, $y=r$

Fix $r=0$, go from $\theta=0$ to $\theta=\pi/2$

$x = 0\cos\theta = 0$
$y = 0\sin\theta = 0$

Fix $r=1$ go from $\underbrace{\theta=0 \text{ to } \theta=\pi/2}$

$x = 1\cos\theta$
$y = 1\sin\theta$ } Traces first quadrant of unit circle.

One-to-one

A transformation T from a region S is one-to-one on S if $T(P)=T(Q)$ only when $P=Q$ where P & Q are points in S.

☒ our example isn't 1-1 (see light green) but it is on interior of S.

Def:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Thm 16.8 (pg. 1075)

Let $T: x = g(u,v)$ $y = h(u,v)$ be a transformation that maps a closed bounded region S in the uv plane to a region R in the xy-plane. Assume T is one-to-one on the interior of S and g & h have cont. first partial derivatives there. If f is cont. on R, then $\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v))\,|J(u,v)|\,dA$

Ex: $x = g(r,\theta) = r\cos\theta$

$y = h(r,\theta) = r\sin\theta$

Jacobian

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$u = r$

$v = \theta$

$$J(r,\theta) = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}$$

$$\cos\theta\,(r\cos\theta) - \cancel{r\sin\theta}\,(-r\sin\theta)\sin\theta$$

$$\cancel{dxdy} = r(\cos^2\theta + \sin^2\theta) = r$$

$$|J(r,\theta)| = r$$

$$\boxed{dxdy = r\,dr\,d\theta}$$

Example 3.

Evaluate:

$$\iint_R \sqrt{2x(y-2x)}\,dA$$

where R is the parallelogram in the x,y plane w/ vertices $(0,0)$, $(0,1)$, $(2,4)$, & $(2,5)$. Use the transformation:

$x = 2u$

$y = 4u + v$

☆ solve for u & v.

$$\boxed{u = \frac{x}{2}}$$

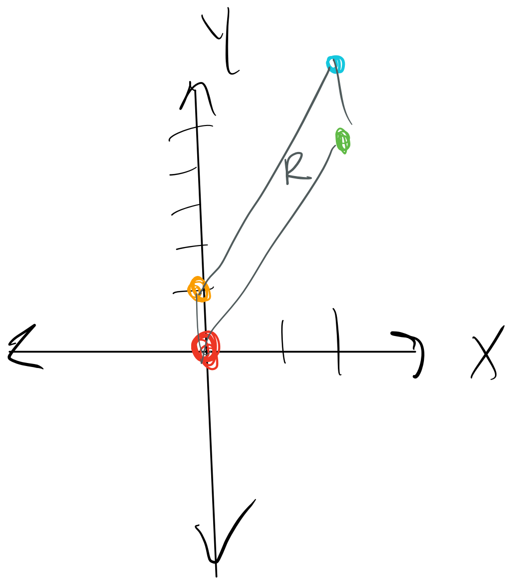
$y = 4u + v$

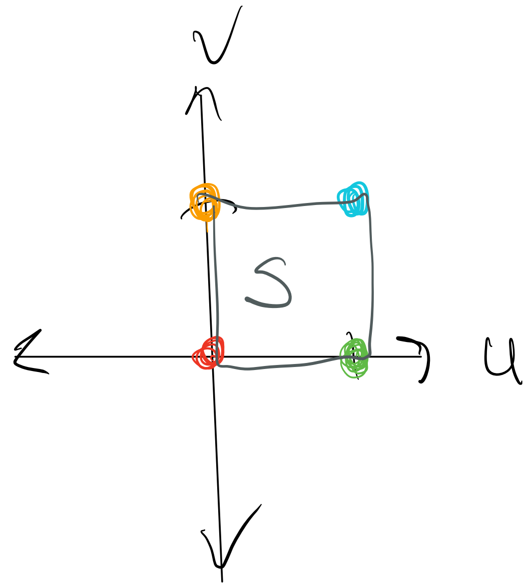
$v = y - 4u$

$v = y - 4\left[\frac{x}{2}\right]$

$$\boxed{v = y - 2x}$$

(x,y)	u	v
$(0,0)$	0	0
$(0,1)$	0	1
$(2,4)$	1	0
$(2,5)$	1	1





Jacobian:
$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

$x = 2u$
$y = 4u + v$

$$\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2\cdot 1 - 0\cdot 4 = 2$$

So by thm...

$$\iint_R \sqrt{2x(y-2x)}\,dA = \iint_S \sqrt{4u(4u+v-4u)}\,|2|\,du\,dv$$

$u = 1$

$$= \int_{u=0} \int_{v=0}^{v=1} \sqrt{4uv} \cdot 2 \, du \, dv$$

$$= 4 \int_{u=0}^{u=1} \int_{v=0}^{v=1} u^{1/2} v^{1/2} \, du \, dv$$

$$= 4 \int_{u=0}^{u=1} u^{1/2} \underbrace{\left[\frac{2}{3} v^{3/2} \Big|_{v=0}^{v=1} \right]}_{2/3} du$$

$$= 4\left(\frac{2}{3}\right) \int_{u=0}^{u=1} u^{1/2} \, du$$

$$= 4\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \boxed{16/9}$$

End Ex: 3