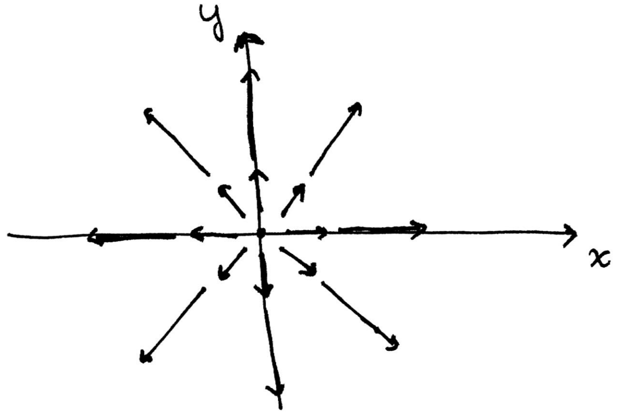
15.1 Vector Fields

Def Let f and g be defined on a region R of $\mathbb{R}^2$. A vector field in $\mathbb{R}^2$ is a function F that assigns to each point in R a vector $\langle f(x,y), g(x,y)\rangle$.

$$F(x,y) = \langle f(x,y), g(x,y)\rangle$$

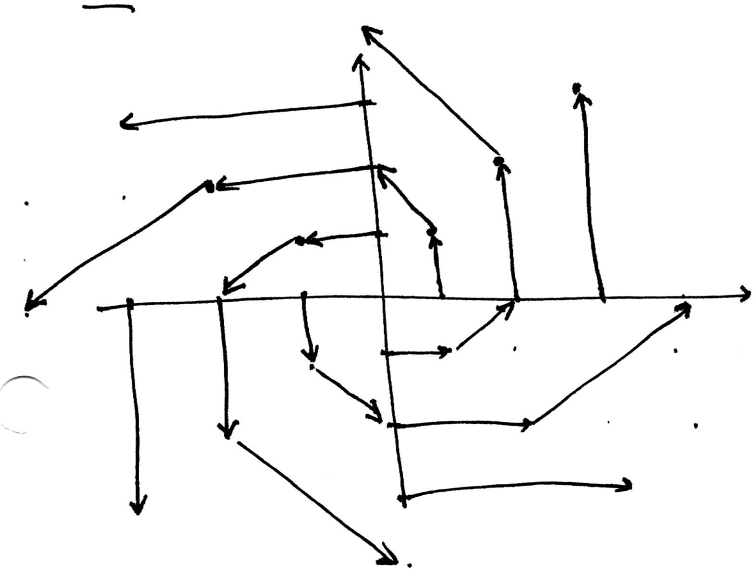
Ex. $F(x,y) = \langle 2x, 2y\rangle$



radial vector field (its vectors point radially away from the origin, parallel to the position vector)

Vector fields are sometimes called flows. Continuous curves that are aligned with the vector field are called streamlines or flow curves.

Ex $F(x,y) = \langle -y, x\rangle$ (a rotation field)



Def. Radial vector Fields in $\mathbb{R}^2$

Let $\mathbf{r} = \langle x, y \rangle$. A vector field of the form $\mathbf{F} = f(x,y)\,\mathbf{r}$, where f is a scalar-valued function, is a radial vector field.

Of specific interest are the radial vector fields

$$\mathbf{F}(x,y) = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y \rangle}{|\mathbf{r}|^p}$$

where p is a real number. At every point (except the origin), the vectors of this field are directed outward from the origin with a magnitude of $|\mathbf{F}| = \dfrac{1}{|\mathbf{r}|^{p-1}}$

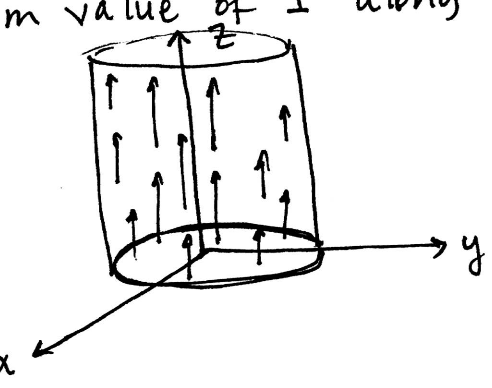
In a similar way, we define vector fields in $\mathbb{R}^3$ and radial vector fields in $\mathbb{R}^3$.

vector field in $\mathbb{R}^3$: $\mathbf{F}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$

and of particular importance the radial vector fields: $\mathbf{F}(x,y,z) = \dfrac{\langle x, y, z \rangle}{|\mathbf{r}|^p}$

Ex vector field: $\mathbf{F}(x,y,z) = \langle 0, 0, 1 - x^2 - y^2 \rangle$, for $x^2 + y^2 \le 1$

For points on the cylinder $x^2 + y^2 = 1$, $F = \langle 0, 0, 0 \rangle$ and there is no motion. In the interior of the cylinder $x^2 + y^2 < 1$ the z-component increases from 0 on the boundary of the cylinder to a maximum value of 1 along the centerline of the cylinder.



Gradient fields and Potential Functions.

One way to generate a vector field is to start with a differentiable scalar-valued function φ, take its gradient and let $F = \nabla\varphi$.

A vector field defined as the gradient of a scalar-valued function φ is called a gradient field and the function φ is called a potential function.

Ex Let $\varphi(x,y) = x^2y^3$

$\Rightarrow \nabla\varphi = \langle 2xy^3, 3x^2y^2 \rangle$

Let $F = \nabla\varphi$

- F is called gradient field
- φ is a potential function for F

Level curves of a potential function are called equipotential curves (curves on which the potential function is constant)

Because the equipotential curves are level curves of φ, the vector field $F = \nabla\varphi$ is everywhere orthogonal to the equipotential curves. Therefore, the vector field is visualized by drawing continuous flow curves that are everywhere orthogonal to the equipotential curves.