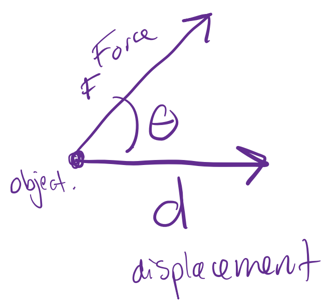
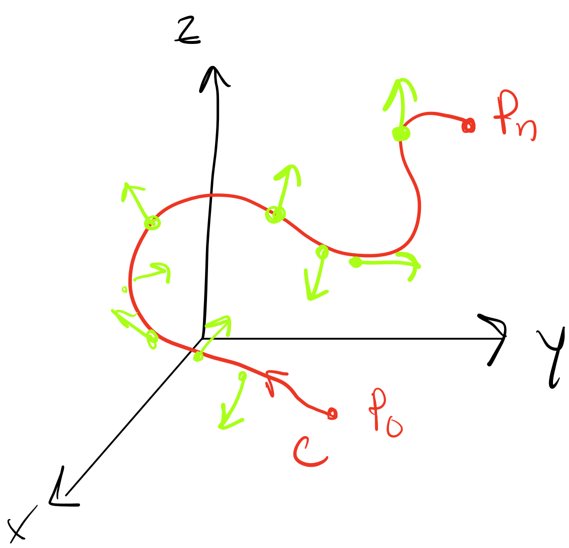
Line integrals of vector fields

Recall from section 13.3 that the work done by a constant force F in moving an object producing a displacement d was defined by

$$W = |F||d| \cos\theta = F \cdot d$$





Suppose **F** is a continuous force field (3D vector field)

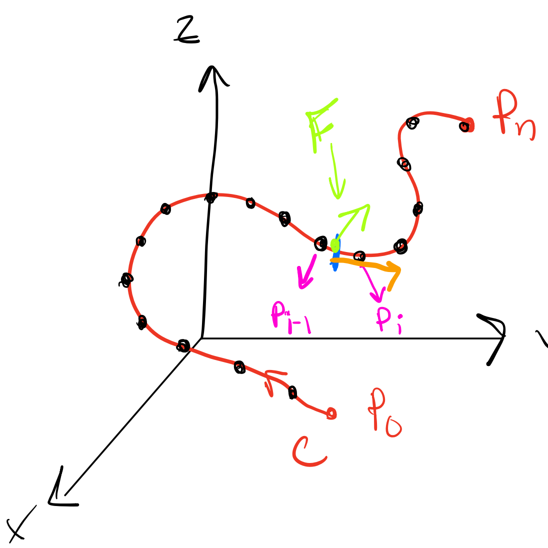
Ex: $F = \langle 3x^2y, 2xz, 4yz \rangle$

*Something that assigns each point in $\mathbb{R}^3$ a vector.

Suppose C is a smooth curve given by

$r(t) = \langle x(t), y(t), z(t) \rangle$

We could ask the question: what is the work done by this force in moving a particle along a smooth curve C.



We will first divid C into small subarcs with length Δs_i. Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ that corresponds to the parameter value t_i^*

$r(t_i^*) = \langle x(t_i^*), y(t_i^*), z(t_i^*) \rangle$

when Δs_i is small, as this particle moves from P_{i-1} to P_i along the curve, it moves approximately in the direction $T(t_i^*)$ (the unit tangent vector at $r(t_i^*)$).

Therefore, the work being done by the force F in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \Delta s_i \mathbf{T}(t_i^*)$$

$\Rightarrow$ the total work done in moving this particle along C is $\approx$

$$\sum_{i=1}^{n} \left[\mathbf{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \mathbf{T}(x(t_i^*), y(t_i^*), z(t_i^*)) \right] \Delta s_i$$

Therefore, we define the <u>work</u> W done by the force field as

$$\boxed{W = \int_C \mathbf{F} \cdot \mathbf{T} \, ds}$$

Rewrite

Recall $T = \dfrac{r'(t)}{|r'(t)|}$

and Recall $ds = |r'(t)|dt$

$\longrightarrow$ From Wednesday

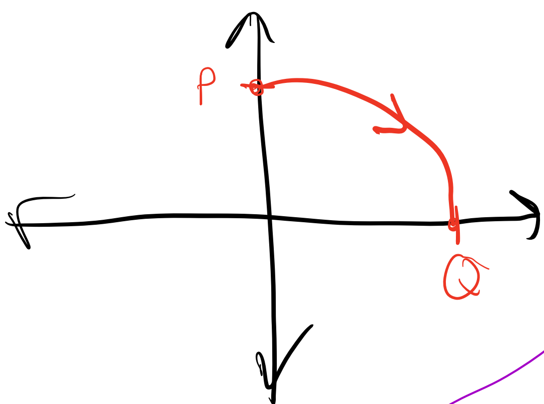
$\Rightarrow$

$$W = \int_{t=a}^{t=b} F(r(t)) \cdot \frac{r'(t)}{\cancel{|r'(t)|}} \cancel{|r'(t)|} dt$$

$$\boxed{W = \int_{t=a}^{t=b} F(r(t)) \cdot r'(t)\, dt}$$

☆ Same formula for 2D or 3D.

Ex: Evaluate $\int_C F \cdot T \, ds$ with $F = \langle y-x, x \rangle$ on the following oriented path in $\mathbb{R}^2$: the quarter circle C from $P = (0,1)$ to $Q = (1,0)$



Parametric equation of C

There are many different ways to describe C parametrically. I will choose

at $t = 0 \longrightarrow (0,1) = P$

$t = \pi/2 \longrightarrow (1,0) = Q$

one way to do this is

$$C = r(t) = \langle \sin t, \cos t \rangle$$

$$0 \le t \le \pi/2$$

(you can check some inbetween values of t to see it goes over the curve C)

$$\text{So} \int_C F \cdot T \, ds = \int_{t=a}^{t=b} F(r(t)) \cdot r'(t) \, dt$$

$$\Rightarrow \int_{t=0}^{t=\pi/2} \langle \cos t - \sin t, \sin t \rangle \cdot \langle \cos t, -\sin t \rangle \, dt$$

$$= \cos 2t$$

$$= \int_{t=0}^{t=\pi/2} \left(\cos^2 t - \underbrace{\sin t \cos t}_{= \frac{1}{2} \sin 2t} + -\sin^2 t \right) dt$$

$\pi/2$

$$= \int_{t=0}^{t=\pi/2} \cos 2t - \frac{1}{2}\sin 2t \, dt$$

$$= \left. \frac{\sin 2t}{2} + \frac{1}{2}\frac{\cos 2t}{2} \right|_{t=0}^{t=\pi/2}$$

$$= \left[0 + \frac{-1}{4}\right] - \left[0 + \frac{1}{4}\right]$$

$$= -\frac{1}{4} - \frac{1}{4} = \boxed{-\frac{1}{2}}$$

☆ If you did along the same path but from Q to P, you would get $\frac{1}{2}$

$$\boxed{\int_{-C} F\cdot T\,ds = -\int_C F\cdot T\,ds}$$

One more example

Evaluate $\displaystyle\int_C F\cdot T\,ds$

for $F = \langle x, y\rangle$ on

$$r(t) = \langle \underset{x}{4t}, \underset{y}{t^2}\rangle$$

for $0 \le t \le 1$

$$\int_C F \cdot T\,ds = \int_{t=a}^{t=b} \underline{F(r(t))} \cdot \underline{r'(t)}\,dt$$

$$= \int_{t=0}^{t=1} \langle \underline{4t, t^2} \rangle \cdot \langle 4, 2t \rangle\,dt$$

$$= \int_{t=0}^{t=1} 16t + 2t^3\,dt$$

$$= 8t^2 + \frac{1}{2}t^4 \Big|_{t=0}^{t=1}$$

$$= 8 + \frac{1}{2} = \boxed{\frac{17}{2}}.$$

Another example:

Evaluate

$F = \langle y, x \rangle$ on the line

Segment from $(1,1)$ to $(5,10)$

$C = r(t) = \langle 1, 1 \rangle + \langle 4, 9 \rangle t$

$= \langle 1 + \overset{x}{4t}, 1 + \overset{y}{9t} \rangle$

$r'(t) = \langle 4, 9 \rangle$

from

$\underbrace{t=0}_{\text{at } (1,1)}$ to $\underbrace{t=1}_{\text{at } (5,10}$

$$\int_C F \cdot T\,ds = \int_{t=a}^{t=b} \underline{F(r(t))} \cdot \underline{r'(t)}\,dt$$

$\langle y, x \rangle$

$$\Rightarrow \int_{t=0}^{t=1} \langle 1+9t,\ 1+4t \rangle \cdot \langle 4, 9 \rangle\,dt$$

$$\Rightarrow \int_{t=0}^{t=1} 4+36t + 9+36t\,dt$$

$$= \int_{t=0}^{t=1} 13 + 72t\,dt$$

$$13t + \frac{72t^2}{2} \Big|_{t=0}^{t=1}$$

$$= 13 + \frac{72}{2}$$

$$= 13 + 36 = \boxed{49}$$