Circulation:

Recall our line integral we used to calculate work done in a force field.

$$W = \int_C F \cdot T \, ds$$

↓ Rewritten in last notes

$$W = \int_{t=a}^{t=b} F(r(t)) \cdot r'(t) \, dt$$

$F(r(t))$ → Force field

$\int_C$ → moving an object along C

The definition of circulation looks really similar.

Def: Let F be a continuous vector field on a region D of $\mathbb{R}^3$ and let C be a closed smooth oriented curve in D. The <u>circulation</u> of F on C is $\int_C F \cdot T \, ds$ where T is the unit vector tangent to C consistent with the orientation.

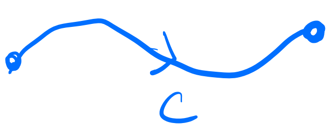
This looks almost exactly the same as our work integral, but I wanted to point out some of the differences.

Work

$$W = \int_C F \cdot T \, ds$$

over here, this has to be a vector field representing force

C is a smooth oriented curve



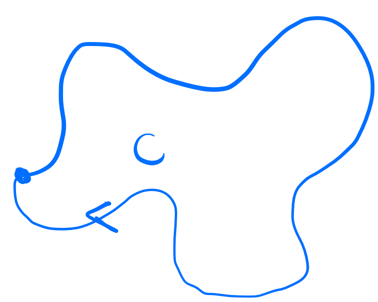
Circulation

$$Cir = \int_C F \cdot T \, ds$$

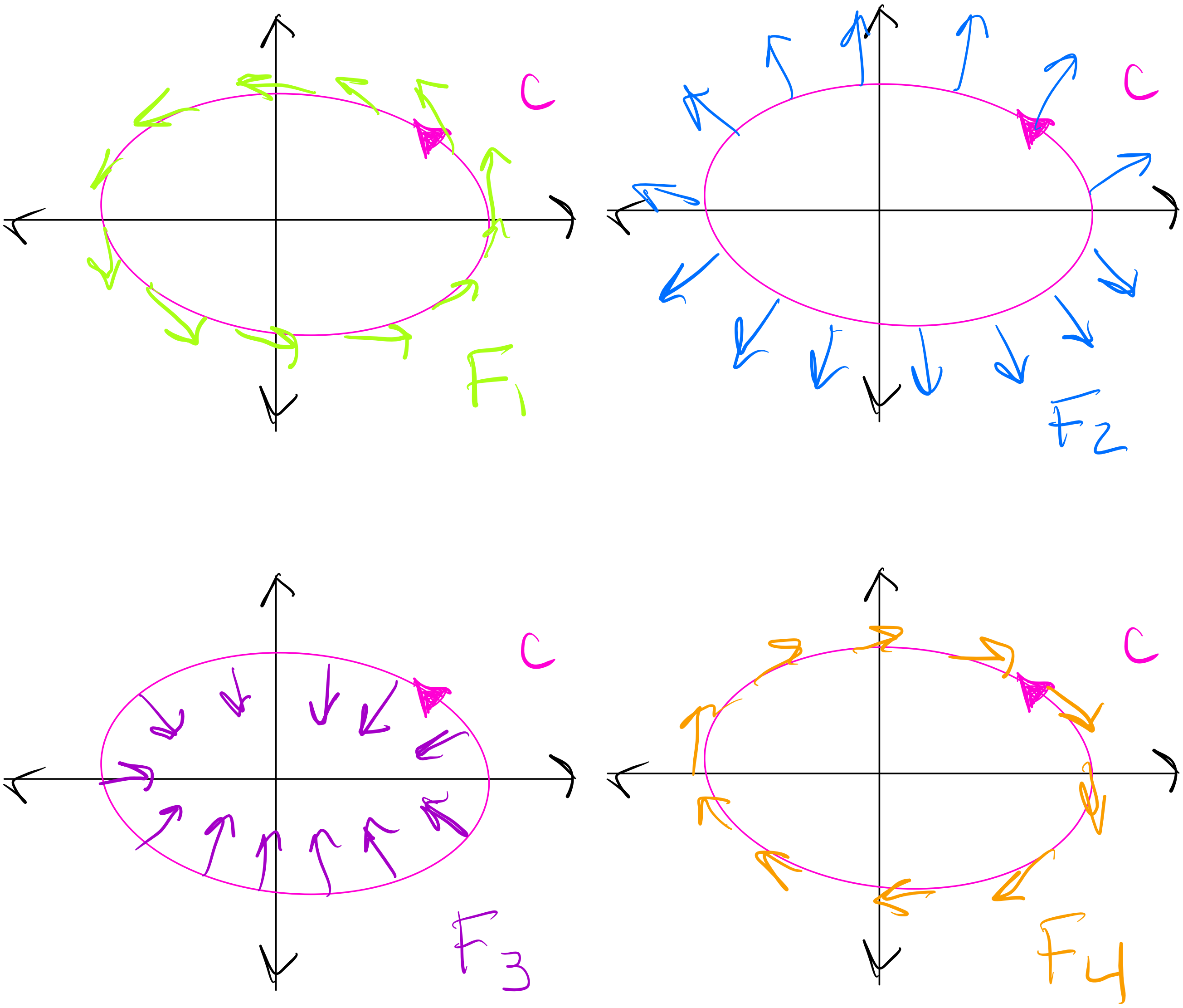
This can just be a general continuous vector field

Here, C is a closed smooth oriented curve

starts & ends at same point



Circulation of F along C is a measure of "how much the vectorfield points in the direction of C.



Determine whether the following are pos, neg, or 0.

$u \cdot v = |u||v| \cos\theta$

$\theta = 0$

1. $\int_C F_1 \cdot T\,ds \Rightarrow$ pos

$\theta = 90$

2. $\int_C F_2 \cdot T\,ds \Rightarrow 0$

$\theta = 90$

3. $\int_C F_3 \cdot T\,ds \Rightarrow 0$

$\theta = \pi$ →

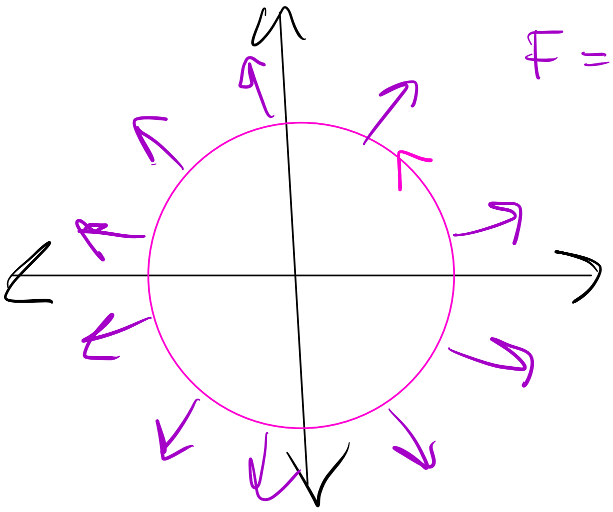
④ $\int_C F_4 \cdot T\,ds \Rightarrow$ neg.

Two circulation computations

Ex: Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields

a. $F = \langle x, y \rangle$ (radial)

b. $F = \langle -y, x \rangle$ (rotational)

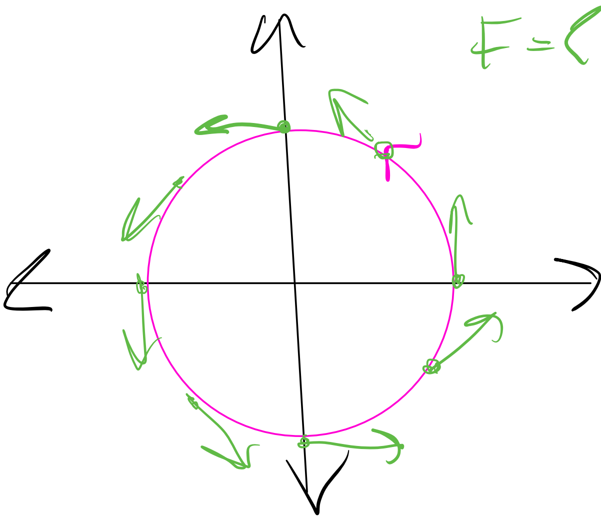


$F = \langle x, y \rangle$

✡ Expect 0

$C = \langle \cos t, \sin t \rangle$ (x = cos t, y = sin t)

$0 \le t \le 2\pi$



$F = \langle -y, x \rangle$

✡ Expect positive.

ⓐ $\text{Cir} = \int_C F \cdot T\, ds = \int_{t=0}^{t=2\pi} F(r(t)) \cdot r'(t) \cdot dt$

using rewrite from last notes.

$\int_{t=0}^{t=2\pi}$

$$= \int_{t=0} \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{t=0}^{t=2\pi} -\cos t \sin t + \sin t \cos t \, dt$$

$$= \int_{t=0}^{t=2\pi} 0 \, dt = \boxed{0}$$

(b.) $$\text{Cir} = \int_C F \cdot T \, ds = \int_{t=0}^{t=2\pi} F(r(t)) \cdot r'(t) \cdot dt$$

$$= \int_{t=0}^{t=2\pi} \langle \overset{-y}{-\sin t}, \overset{x}{\cos t} \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{t=0}^{t=2\pi} \sin^2 t + \cos^2 t \, dt$$

$$= \int_{t=0}^{t=2\pi} 1 \, dt = \boxed{2\pi}$$

☆ Same formula & idea for 3D circulation