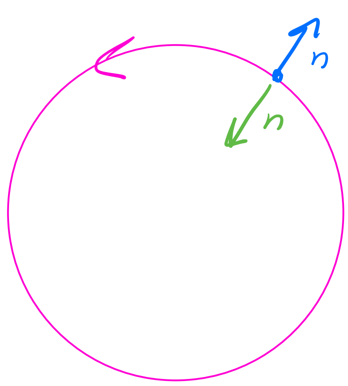
Flux

Assume $F = \langle f, g \rangle$ is a continuous vector field on a region R of $\mathbb{R}^2$. We let C be a smooth oriented curve in R that doesn't intersect itself, (C may or may not be closed).

To compute the flux, of the vector field across C, we "add up" the components of F orthogonal or normal to C at each point of C.

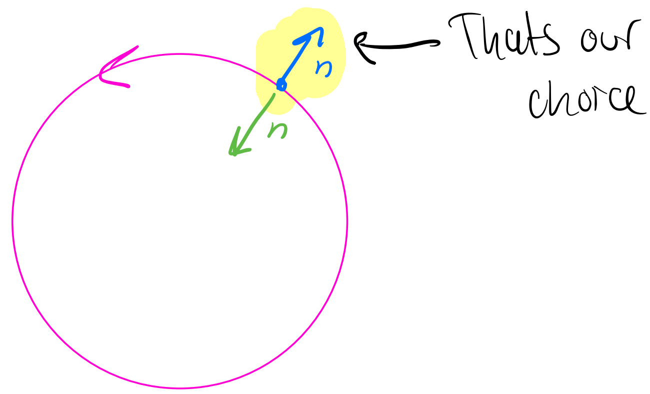
* there are two choices of normal vector.



two choices.

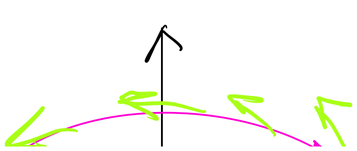
So we need to define which one we will use.

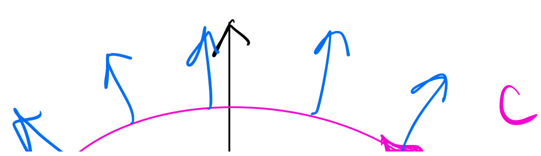
* In the 2D case, when we have a closed curve, we will take n to be the normal vector pointing outward.

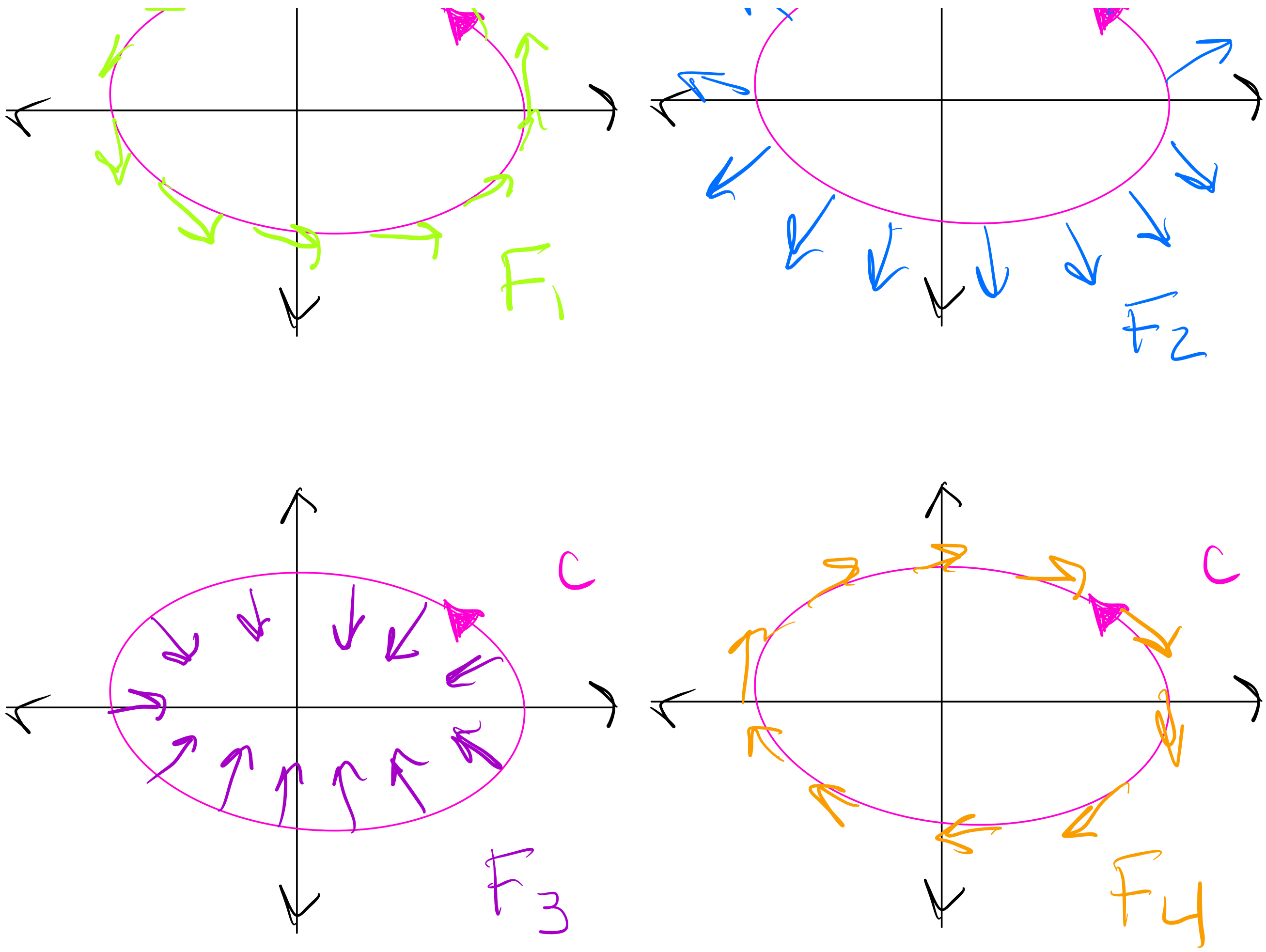


$$\text{Flux} = \int_C F \cdot n \, dS$$

Ex: Estimate if the flux positive, 0, or negative across C for the following 4 vector fields.







① $\int_C F_1 \cdot n \, ds \Rightarrow \theta = \pi/2$

0

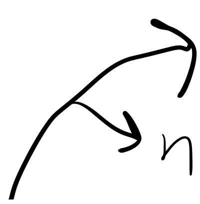
② $\int_C F_2 \cdot n \, ds \Rightarrow \theta = 0$

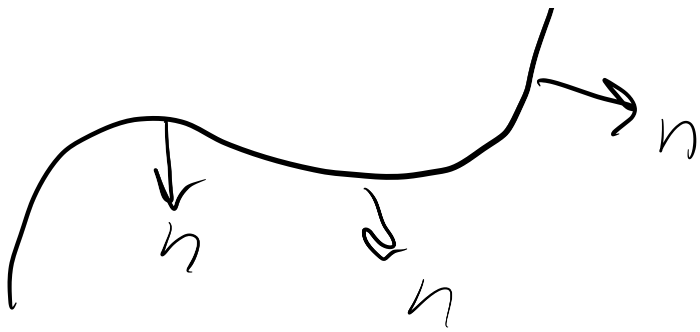
pos

3. $\int_C F_3 \cdot n \, ds \Rightarrow \theta = \pi$ neg

4. $\int_C F_4 \cdot n \, ds \Rightarrow \theta = \pi/2$ neg.

If C is not a closed curve, then the unit normal vector points to the right as the curve moves in the positive direction.

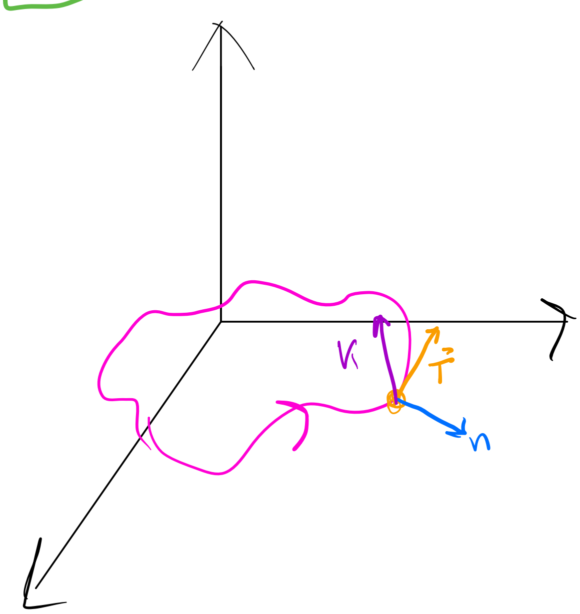


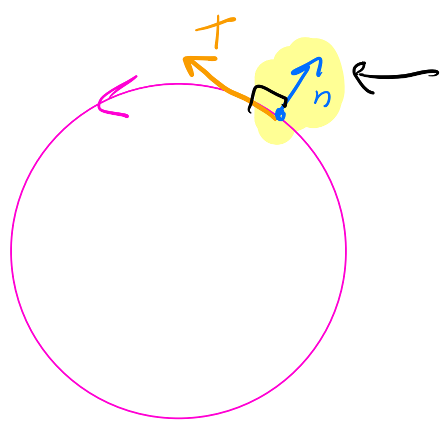


How do we actually compute

$$\text{Flux} = \int_C F \cdot n \, ds$$

$\boxed{\text{We need a formula for } n.}$





$K = \langle 0,0,1 \rangle$

$$C = r(t) = \langle x(t), y(t) \rangle$$

$$T = \frac{r'(t)}{|r'(t)|} = \frac{\langle x'(t), y'(t) \rangle}{|r'(t)|}$$

$$T_x = \frac{x'(t)}{|r'(t)|} \qquad T_y = \frac{y'(t)}{|r'(t)|}$$

$$n = T \times K = \begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,1) \\ \uparrow & \uparrow & \uparrow \end{array}$$

$$n = T \times K = \begin{vmatrix} i & j & K \\ T_x & T_y & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} T_y & 0 \\ 0 & 1 \end{vmatrix} i - \begin{vmatrix} T_x & 0 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} T_x & T_y \\ 0 & 0 \end{vmatrix} K$$

$$= T_y i - T_x j$$

$$= \frac{y'(t)}{|r'(t)|}(1,0,0) - \frac{x'(t)}{|r'(t)|}(0,1,0)$$

$$= \left(\frac{y'(t)}{|r'(t)|}, -\frac{x'(t)}{|r'(t)|}, 0 \right)$$

viewed in xy plane

$$\boxed{n = \left(\frac{y'(t)}{|r'(t)|}, -\frac{x'(t)}{|r'(t)|} \right)}$$

$$\text{Flux} = \int_C F \cdot n \, dS$$

Recall

$dS = \frac{ds}{dt} dt =$

$= s'(t)\, dt$

$= |r'(t)|\, dt$

$$= \int_C F \cdot \frac{\langle y'(t), -x'(t) \rangle}{|r'(t)|} |r'(t)|\, dt$$

$$\boxed{= \int_C F \cdot \langle y'(t), -x'(t) \rangle \, dt \quad \text{Flux}}$$

$$\text{Flux} = \int_C F \cdot n \, dS = \int_C F \cdot \langle y'(t), -x'(t) \rangle \, dt$$

* called outward flux bc choice of n *

Lets calculate Flux

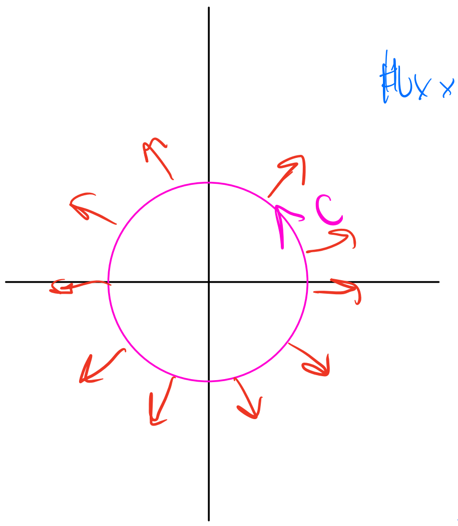
Ex: Find the outward flux across the unit circle with counter clockwise orientation for the following vector fields.

$\circlearrowleft C$

a. $F = \langle x, y \rangle$ radial

$C = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle = r(t)$

$0 \le t \le 2\pi$



Flux $\int_C F \cdot n \, dS = \int_C F \cdot \langle y'(t), -x'(t) \rangle dt$

$\int_{t=0}^{t=2\pi} \langle \cos t, \sin t \rangle \cdot \langle \cos t, \sin t \rangle dt$

$\int_{t=0}^{t=2\pi} \cos^2 t + \sin^2 t \, dt$

$\int_{t=0}^{t=2\pi} 1 \, dt = 2\pi$

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b. $F = \langle -y, x \rangle$ along same C

$C = r(t) = \langle \overset{x(t)}{\cos(t)}, \overset{y(t)}{\sin(t)} \rangle$

$\text{Flux} = \int_{t=0}^{t=2\pi} \langle -y(t), x(t) \rangle \cdot \langle y'(t), -x'(t) \rangle dt$

$= \int_{t=0}^{t=2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle \cos t, \sin(t) \rangle dt$

$$= \int_{t=0}^{t=2\pi} -\sin t \cos t + \cos t \sin t \, dt$$

$$= \boxed{0}$$