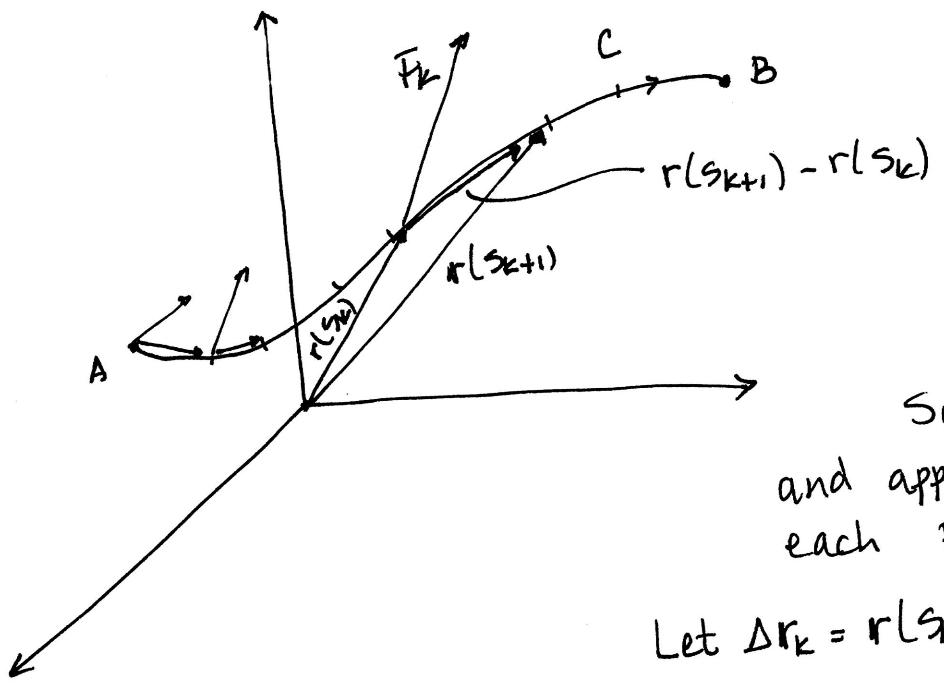
15.2 Line Integrals

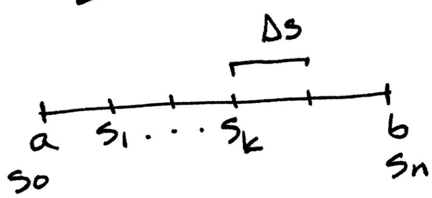
Let C be a curve in $\mathbb{R}^3$ parameterized by arc length.

$C: \mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$ $\quad a \le s \le b$



Let $F(x, y, z)$ be a variable force field. We want to compute the work done in moving an object along the curve C.

Subdivide C in n-pieces and approximate the work in each piece.



Let $\Delta \mathbf{r}_k = \mathbf{r}(s_{k+1}) - \mathbf{r}(s_k)$, then

$$w_k = F_k \cdot \Delta \mathbf{r}_k$$

The total work can be approximated by adding all these w_k's.

$$W \simeq \sum_{k=1}^{n} F_k \cdot \Delta \mathbf{r}_k$$

So, we define the total work W as the limit of this sum as $n \to \infty$, or equivalently as $\Delta s \to 0$.

$$\Delta \mathbf{r}_k = \mathbf{r}(s_{k+1}) - \mathbf{r}(s_k) = \mathbf{r}(s_k + \Delta s) - \mathbf{r}(s_k) =$$

$$= \left[\frac{\mathbf{r}(s_k + \Delta s) - \mathbf{r}(s_k)}{\Delta s} \right] \Delta s$$

$\therefore$ as $\Delta s \to 0$ $\qquad \Delta \mathbf{r}_k \to \left(\frac{d\mathbf{r}}{ds} \right) ds$

Since $\mathbf{r}$ is parameterized by arc length,

$\frac{d\mathbf{r}}{ds}$ is a unit tangent vector T

$$\therefore \quad W = \lim_{n\to\infty} \sum_{k=1}^{n} F_k \cdot \Delta r_k \quad \text{can be written as:}$$

$$W = \int_C F \cdot d\mathbf{r} = \int_C F \cdot \left(\frac{d\mathbf{r}}{ds}\right) ds = \int_C F \cdot T \, ds$$

<u>Def</u> Let F be a force field, and let $C: \mathbf{r}(s) = \langle x(s), y(s), z(s) \rangle$ $a \le s \le b$ be a smooth curve with a unit tangent vector T consistent with the orientation. The work done in moving an object along C in the positive direction is:

$$W = \int_C F \cdot T \, ds$$

If C is not parameterized by arc length, say $C: \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then since $s = \int_a^t |\mathbf{r}'(u)| \, du$

$\frac{ds}{dt} = |\mathbf{r}'(t)| \Rightarrow ds = |\mathbf{r}'(t)| dt$, then

$$W = \int_C F \cdot T \, ds = \int_a^b F \cdot \underbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}_{T} \underbrace{|\mathbf{r}'(t)| \, dt}_{ds}$$

$$\therefore \quad W = \int_a^b F \cdot \mathbf{r}'(t) \, dt$$ //

if $r(t) = \langle x(t),\ y(t),\ z(t) \rangle$

$$r'(t)dt = \langle x'(t),\ y'(t),\ z'(t) \rangle\, dt$$
$$= \langle x'(t)\,dt,\ y'(t)\,dt,\ z'(t)\,dt \rangle$$
$$= \langle dx,\ dy,\ dz \rangle$$

and $r'(t)dt = dr$

$$\therefore\ \boxed{W = \int_a^b F \cdot dr \quad \text{where } dr = \langle dx, dy, dz \rangle}$$

Ex Given the force field $F = \langle -y, x, z \rangle$ on the helix $r(t) = \langle 2\cos t,\ 2\sin t,\ \frac{t}{2\pi} \rangle$ $\quad 0 \le t \le 2\pi$

find the work required to move an object on this curve.

$$F = \langle -y, x, z \rangle = \langle -2\sin t,\ 2\cos t,\ \tfrac{t}{2\pi} \rangle$$

$$dr = \langle dx, dy, dz \rangle = \langle -2\sin t\, dt,\ 2\cos t\, dt,\ \tfrac{1}{2\pi}\, dt \rangle$$

$$F \cdot dr = 4\sin^2 t\, dt + 4\cos^2 t\, dt + \frac{t}{4\pi^2}\, dt$$

$$= \left(4 + \frac{t}{4\pi^2}\right) dt$$

$$\therefore\ W = \int_0^{2\pi} \left(4 + \frac{t}{4\pi^2}\right) dt = 4t + \frac{t^2}{8\pi^2}\Big|_0^{2\pi} = 8\pi + \frac{4\pi^2}{8\pi^2}$$

$$= \underline{8\pi + \tfrac{1}{2}}$$

Circulation. Another example of line integrals of vector fields.

Let C be a closed curve. The circulation of F along C is a measure of how much of the vector field points in the direction of C.

<u>Def</u> Let F be a vector field and let C be a closed smooth oriented curve. The circulation of F on C is

$$\int_C F \cdot T \, ds \, ,$$ where T is the unit vector tangent

to C consistent with the orientation.

<u>Ex</u> Let C be the unit circle with counterclockwise orientation. Find the circulation on C of the following vector fields.

(a) The radial vector field
$F = \langle x, y \rangle$

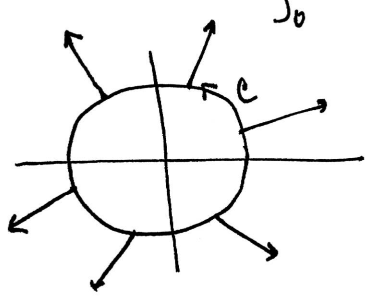
$C: r(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

$\Rightarrow r'(t) = \langle -\sin t, \cos t \rangle$

$F = \langle x, y \rangle = \langle \cos t, \sin t \rangle$

$F \cdot dr = \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$
$= (-\sin t \cos t + \sin t \cos t) dt$
$= 0$

$\therefore$ circulation $= \displaystyle\int_0^{2\pi} F \cdot dr = 0.$

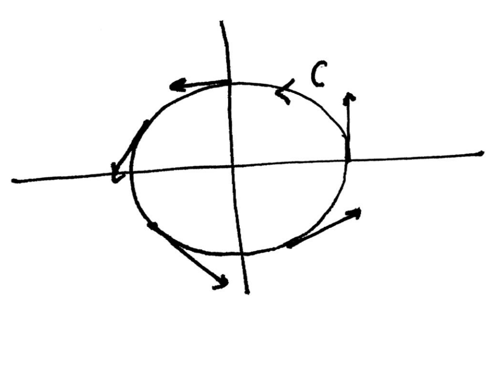


(b) The rotation vector field
$F = \langle -y, x \rangle$

$F \cdot dr = \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$
$= (\sin^2 t + \cos^2 t) \, dt = dt$

Circulation $= \displaystyle\int_0^{2\pi} F \cdot dr = \int_0^{2\pi} dt$

$= \underline{2\pi}$ //



Flux of two-dimensional vector fields.

Another line integral that we will study, is the one when we integrate the component of a vector field but instead of taking the tangential component, we take the normal component:

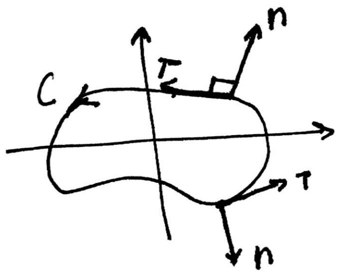
$$\int_C F \cdot n \, ds$$

Lets first define which normal vector we will use.

Some restrictions for the moment:

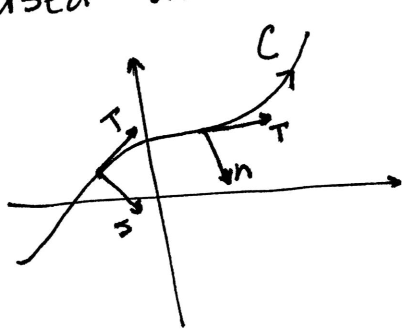
We will for now consider F to be a two-dimensional vector field.

- If C is a closed curve, then n will be the normal vector that points outward along the curve.



n perpendicular to T, with $|n| = 1$

- If C is not a closed curve, the unit normal vector points to the right (when viewed from above) as the curve is traversed in the positive direction.



Let C be a curve not necessarily parameterized by arc length

$C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$

$$\Rightarrow T = \frac{r'(t)}{|r'(t)|} = \left\langle \frac{x'(t)}{|r'(t)|}, \frac{y'(t)}{|r'(t)|} \right\rangle$$

To construct n, Let $T = \left\langle \frac{x'(t)}{|r'(t)|}, \frac{y'(t)}{|r'(t)|}, 0 \right\rangle$

and $\hat{k} = \langle 0, 0, 1 \rangle$

$$\text{Then } n = T \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{x'(t)}{|r'(t)|} & \frac{y'(t)}{|r'(t)|} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|}, 0 \right\rangle$$

we identify n with $\left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|} \right\rangle$

Now, we are ready to compute the integral $\int_C F \cdot n \, ds$:

$$\int_C F \cdot n \, ds = \int F \cdot \left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|} \right\rangle |r'(t)| \, dt$$

$$= \int F \cdot \langle y'(t), -x'(t) \rangle \, dt$$

If $F = \langle f, g \rangle$, then

$$\int_C F \cdot n \, ds = \int_a^b \langle f(t), g(t) \rangle \cdot \langle y'(t), -x'(t) \rangle \, dt$$

$$= \int_a^b (f(t)\, y'(t) - g(t)\, x'(t)) \, dt$$

Def Flux.

The flux of the vector field F across C is

$$\int_C F \cdot n \, ds = \int_a^b \left(f(t)\, y'(t) - g(t)\, x'(t) \right) dt .$$

Ex. Find the flux across the unit circle with counterclockwise orientation for the following vector fields.

(a) The radial vector field
$F = \langle x, y \rangle$

$C: r(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

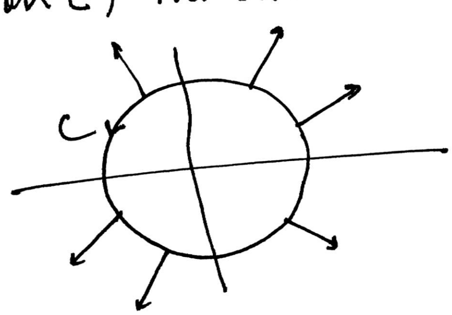
$F = \langle \cos t, \sin t \rangle$

$x(t) = \cos t \Rightarrow x'(t) = -\sin t$
$y(t) = \sin t \Rightarrow y'(t) = \cos t$

$$\int_C F \cdot n \, ds = \int_0^{2\pi} [(\cos t)(\cos t) - (\sin t)(-\sin t)] \, dt$$

$$= \int_0^{2\pi} dt = \underline{2\pi}$$

radial field is aligned with normal vectors on C, the outward flux is positive

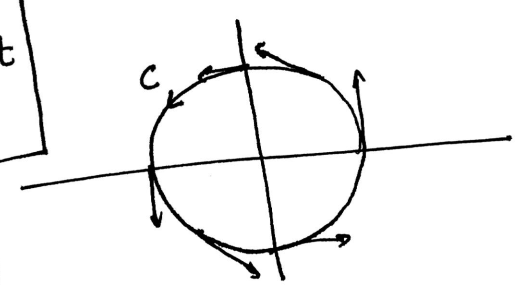


(b) The rotation vector field
$F = \langle -y, x \rangle$

$F = \langle -\sin t, \cos t \rangle$

$$\int_C F \cdot n \, ds = \int_0^{2\pi} [(-\sin t)(\cos t) - \cos t (-\sin t)] \, dt$$

$$= 0$$

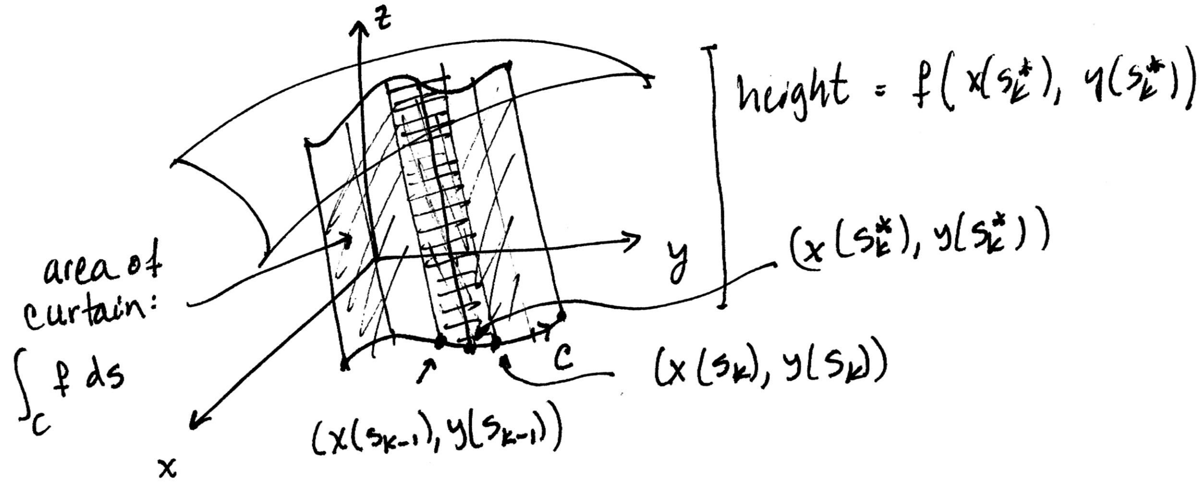


The rotation field is orthogonal to n at all points of C, the outward flux across C is zero.

One more line integral we will consider.

Instead of a vector field, lets consider a scalar field.

Let $z = f(x,y)$.



<u>Def.</u> Suppose the scalar-valued function f is defined on the smooth curve $C: \mathbf{r}(s) = \langle x(s), y(s) \rangle$, parameterized by arc length. The line integral of f over C is:

$$\int_C f(x(s), y(s))\,ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*))\,\Delta s_k$$

provided this limit exists over all partitions of C. When the limit exists, f is said to be integrable on C.

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if $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$

$$\Rightarrow \int_C f\,ds = \int_a^b f(x(t), y(t))\,|\mathbf{r}'(t)|\,dt$$