15.3 Conservative vector fields

Def A vector field F is said to be conservative on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if $\exists\ \varphi$ such that $F = \nabla\varphi$ on that region.

Thm Test for conservative vector fields
Let $F = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region D of $\mathbb{R}^3$, where f, g, and h have continuous first partial derivatives on D. Then F is a conservative vector field on D if and only if:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Ex. Let $F = \langle x, -y \rangle$ $\qquad \frac{\partial x}{\partial y} = 0, \quad \frac{\partial(-y)}{\partial x} = 0$

$\therefore$ F is conservative

Ex. Let $F = \langle -y, x \rangle$ $\qquad \frac{\partial(-y)}{\partial y} = -1, \quad \frac{\partial x}{\partial x} = 1$

$-1 \neq 1$ $\qquad \therefore$ F is not conservative

(F is not the gradient of some function)

Finding potential functions

$F = \langle x, -y \rangle$ is conservative, that means that

$\exists\, \varphi(x,y)$ such that $F = \nabla \varphi$.

Lets find this potential function φ.

$$F(x,y) = \nabla \varphi(x,y) = \langle \varphi_x(x,y), \varphi_y(x,y) \rangle = \langle x, -y \rangle$$

$$\varphi_x(x,y) = x \Rightarrow \varphi(x,y) = \int \varphi_x(x,y)\,dx = \int x\,dx$$

$$= \frac{x^2}{2} + f(y)$$

If $\varphi(x,y) = \frac{x^2}{2} + f(y) \Rightarrow \varphi_y(x,y) = f'(y)$

$\therefore$ since $\varphi_y(x,y) = -y \Rightarrow f'(y) = -y$

$\therefore f(y) = -\frac{y^2}{2} + K$

$\therefore \varphi(x,y) = \frac{x^2}{2} - \frac{y^2}{2} + K$

If we want just one potential function, it is customary to take the one with $K=0$

$$\therefore \varphi(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$$

Finding potential functions

Ex. Let $F = \langle y+z,\ x+z,\ x+y \rangle$

Show that F is a conservative field.

Let $f(x,y,z) = y+z$, $\quad g(x,y,z) = x+z$, $\quad h(x,y,z) = x+y$

$\frac{\partial f}{\partial y} = 1 \qquad \frac{\partial g}{\partial x} = 1 \qquad \therefore \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ ✓

$\frac{\partial f}{\partial z} = 1 \qquad \frac{\partial h}{\partial x} = 1 \qquad \therefore \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$ ✓

$\frac{\partial g}{\partial z} = 1 \qquad \frac{\partial h}{\partial y} = 1 \qquad \therefore \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$ ✓

$\therefore$ F is conservative.

Find a potential function for F. $\quad F = \nabla \varphi$.

$\nabla \varphi = \langle \varphi_x(x,y,z), \varphi_y(x,y,z), \varphi_z(x,y,z) \rangle = \langle y+z, x+z, x+y \rangle$

$\varphi_x = y+z \Rightarrow \varphi(x,y,z) = \int (y+z)\,dx = xy + xz + G(y,z)$

$\Rightarrow \varphi_y = x + \frac{\partial G}{\partial y} = x+z$

$\Rightarrow \frac{\partial G}{\partial y} = z \quad \Rightarrow G(y,z) = \int z\,dy$

$= yz + H(z)$

$\therefore \varphi(x,y,z) = xy + xz + yz + H(z)$

$\varphi_z = x + y + H'(z) = x+y$

$\Rightarrow H'(z) = 0 \quad \Rightarrow H(z) = k.$

$\therefore \varphi(x,y,z) = xy + xz + yz$ $\quad$ (taking $k=0$).

Fundamental Theorem for Line Integrals

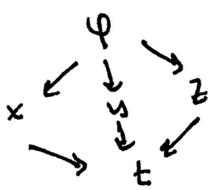
Let R be a region of $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R.

If $\mathbf{F} = \nabla\varphi$, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

for all points A and B in R and all piecewise smooth oriented curves C in R from A to B.

Pf. $\quad C: \ \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle \qquad a \le t \le b$



$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}$$

$$= \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$$

$$= \nabla\varphi\cdot\mathbf{r}'(t)$$

$$= \mathbf{F}\cdot\mathbf{r}'(t)$$

$$\Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b \frac{d\varphi}{dt}\,dt = \varphi(B) - \varphi(A)$$

$$\varphi(x(t), y(t), z(t))\Big|_{t=a}^{t=b}$$

$$= \varphi(x(b), y(b), z(b)) - \varphi(x(a), y(a), z(a))$$

$$= \varphi(B) - \varphi(A).$$

Observations.

If F is conservative, that is if $F = \nabla\varphi$, by the fundamental theorem for line integrals, the line integral just depends on the last point on the curve C and the first point on the curve and not at all on the path.

If the curve C is a closed curve then the initial point and end point are the same and the line integral would be zero. (Of course, this assuming $F = \nabla\varphi$)

Notation for a line integral along a closed curve

$$\oint_C F \cdot dr$$

One can show that all the following are equivalent:

$$\begin{tikzcd}
 & F \text{ is conservative } (F = \nabla\varphi) \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
\text{Path Independence} \arrow[rr, Leftrightarrow] & & \oint_C F \cdot dr = 0
\end{tikzcd}$$