15.4 Green's Theorem.

Thm Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

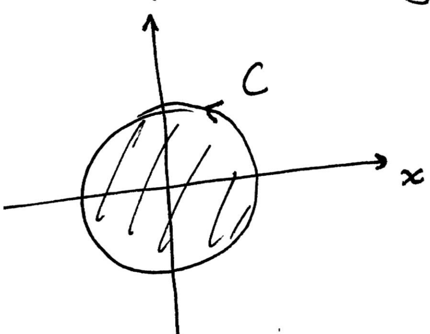
<u>Circulation Form</u>

Def. Curl.

The two dimensional curl of the vector field $F = \langle f, g \rangle$ is $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$.

If the curl is zero throughout a region, the vector field is irrotational on that region.

Ex. Let $F = \langle -y, x \rangle$, and let $C: x^2+y^2 = 1$, unit circle, oriented counterclockwise.
Find $\oint_C F \cdot dr$ using Green's thm.



R is the region enclosed by C.

$R: x^2+y^2 \le 1.$ $\qquad F = \langle -y, x \rangle = \langle f, g \rangle$

$$\frac{\partial g}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = -1$$

$$\therefore \oint_C F \cdot dr = \iint_R (1-(-1))\,dA = \iint_R 2\,dA = 2\,\text{Area}(R) = 2\pi$$

<u>Thm</u> The area of a region R enclosed by a curve C is:

$$\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

<u>Ex.</u> Find the area of the ellipse $x^2 + 4y^2 = 16$ using a line integral.

C: $x^2 + 4y^2 = 16$ $\qquad$ $r(t) = \langle 4\cos t,\ 2\sin t\rangle$ $\quad 0 \le t \le 2\pi$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$ $\qquad$ $x = 4\cos t$ $\qquad$ $dx = -4\sin t\,dt$

$y = 2\sin t$ $\qquad$ $dy = 2\cos t\,dt$

$$\therefore \text{ Area of ellipse } = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (4\cos t)(2\cos t\,dt) - (2\sin t)(-4\sin t\,dt)$$

$$= 4\int_0^{2\pi} dt = \underline{8\pi}$$

<u>Thm</u> Green's Thm. <u>Flux form</u>

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $F = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then:

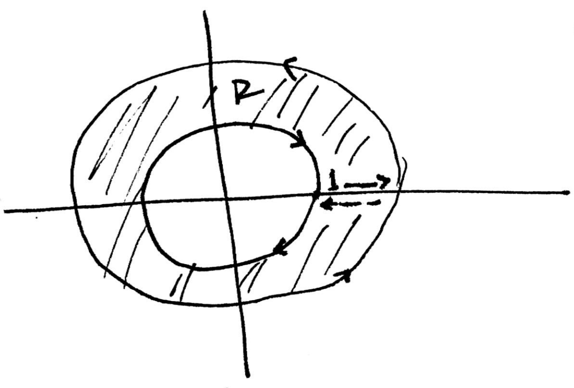
$$\oint_C F\cdot n\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

where n is the outward unit normal vector on the curve.

Def. Divergence.

The two dimensional divergence of the vector field $F = \langle f, g \rangle$ is $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is source free on that region.

Ex. Find the outward flux of the vector field $F = \langle xy^2, x^2 y \rangle$ across the boundary of the annulus
$R = \{ (x,y) \mid 1 \le x^2 + y^2 \le 4 \}$



$$\oint_C F \cdot n \, ds = \oint_C f \, dy - g \, dx =$$

$$= \oint_C xy^2 \, dy - x^2 y \, dx$$

$$= \iint_R (y^2 + x^2) \, dA$$

$$= \int_0^{2\pi} \int_1^2 (r^2) \, r \, dr \, d\theta$$

$$= \frac{15\pi}{2}.$$

Stream functions.

Let $F = \langle f, g \rangle$ be a vector field, a stream function for the vector field -if it exists- is a function ψ that satisfies $\frac{\partial \psi}{\partial y} = f$, $\frac{\partial \psi}{\partial x} = -g$.

Notice that divergence $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$

So, a stream function guarantees that the vector field has zero divergence.

The level curves of a stream function are called flow curves.