15.5 Divergence and Curl

$\mathbb{R}^2$

Let f be a scalar field $f(x,y)$

and let F be a vector field $F(x,y) = \langle f(x,y), g(x,y) \rangle$

we define the del operator as: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$

we can apply the del operator to a scalar field or to a vector field:

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ when applied to a scalar field, we get our known gradient of f encountered before.

We have two ways to apply the del operator to a vector field:

using the dot product: $\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle f, g \rangle$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

using the cross product: $\nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right\rangle \times \langle f, g, 0 \rangle$

$$= \left\langle 0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

... Divergence and Curl

$\mathbb{R}^3$

Similarly, we can extend all these to scalar fields and vector fields in $\mathbb{R}^3$.

Define the del operator: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

When applied to a scalar field $f(x, y, z)$ we get

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ the gradient of f.

When applied to a vector field $F = \langle f, g, h \rangle$ using the dot product:

$$\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

using the cross product:

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

Def Divergence of a vector field

The divergence of a vector field $F = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div } F = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

If $\nabla \cdot F = 0$, the vector field F is said to be source free.

Def Curl of a vector field

The curl of a vector field $F = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{curl } F = \nabla \times F = \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

If $\nabla \times F = \mathbf{0}$, the vector field F is said to be irrotational.

Divergence properties: $\nabla \cdot (F+G) = \nabla \cdot F + \nabla \cdot G$

$$\nabla \cdot (cF) = c(\nabla \cdot F)$$

Curl properties: $\nabla \times (F+G) = (\nabla \times F) + (\nabla \times G)$

$$\nabla \times (cF) = c(\nabla \times F)$$

- The curl of a conservative field:

Let F be a conservative field $\Rightarrow$ $F = \nabla \varphi$

$\Rightarrow F = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

The curl F $= \nabla \times F = \nabla \times (\nabla \varphi)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \hat{i}(\varphi_{yz} - \varphi_{yz}) - \hat{j}(\varphi_{zx} - \varphi_{zx}) + \hat{k}(\varphi_{xy} - \varphi_{xy})$$

$$= \langle 0, 0, 0 \rangle$$

$\therefore$ $\nabla \times (\nabla \varphi) = 0$

The curl of the gradient is the zero vector.

- Divergence of the curl

$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \left(\langle h_y - g_z, f_z - h_x, g_x - f_y \rangle \right)$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

$$= 0 + 0 + 0 = 0$$

The divergence of the curl is 0.