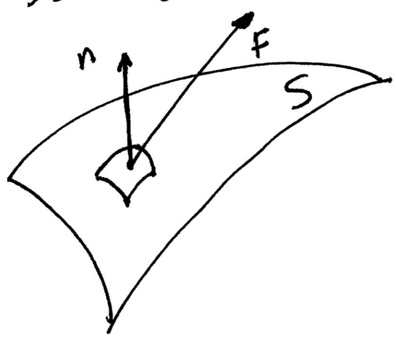
15.6 Surface Integrals.

Surface Integrals of Vector Fields.

Flux integrals.

We aim to compute the net flux of the vector field across the surface.



If the surface is closed n points outward.

If S is not closed, n points upward, (in the direction of the positive z-axis).

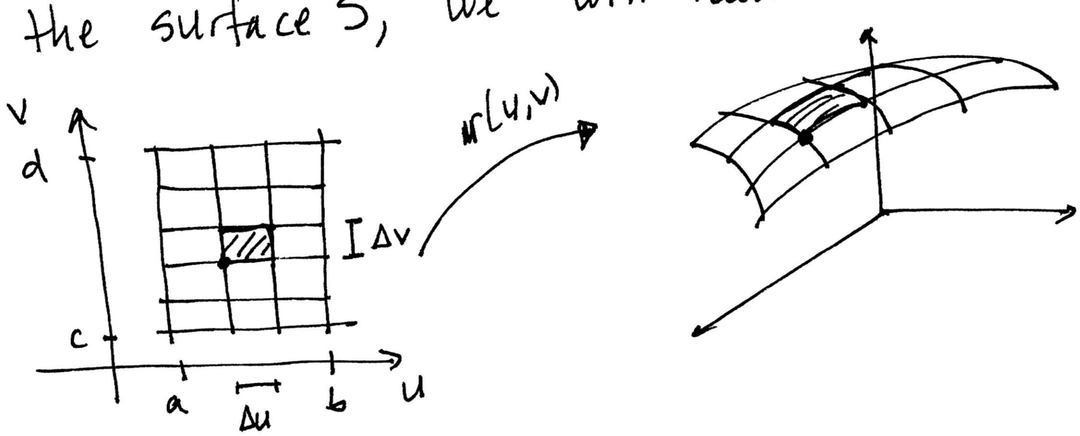
We want to define:

$$\iint_S F \cdot n \, dS$$

where S is the given surface and dS represents the element of surface area.

first lets try to see how we will compute dS.

The first thing we are going to do is parameterize the surface S, we will need two parameters:



Consider a point (u_k, v_k), the image of this point under $\mathbf{r}$ is: $\mathbf{r}(u_k, v_k)$, now, lets measure the change in the u-direction and the v-direction

The point $(u_k + \Delta u, v_k)$ in the surface will be: $\mathbf{r}(u_k+\Delta u, v_k)$

Therefore the change from (u_k, v_k) to $(u_k+\Delta u, v_k)$ in the surface is: $\mathbf{r}(u_k+\Delta u, v_k) - \mathbf{r}(u_k, v_k)$

similarly, the change from (u_k, v_k) to $(u_k, v_k+\Delta v)$ in the surface will be $\mathbf{r}(u_k, v_k+\Delta v) - \mathbf{r}(u_k, v_k)$

when Δu is small $\dfrac{\mathbf{r}(u_k+\Delta u, v_k) - \mathbf{r}(u_k, v_k)}{\Delta u} \simeq \left.\dfrac{\partial \mathbf{r}}{\partial u}\right|_{(u_k, v_k)}$

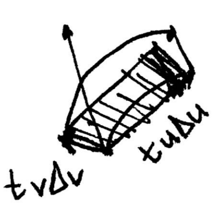
and $\dfrac{\mathbf{r}(u_k, v_k+\Delta v) - \mathbf{r}(u_k, v_k)}{\Delta v} \simeq \left.\dfrac{\partial \mathbf{r}}{\partial v}\right|_{(u_k, v_k)}$

Let $t_u = \dfrac{\partial \mathbf{r}}{\partial u}$ and $t_v = \dfrac{\partial \mathbf{r}}{\partial v}$ $\Rightarrow$ $\begin{array}{l} \mathbf{r}(u_k+\Delta u, v_k) - \mathbf{r}(u_k, v_k) \simeq t_u \Delta u \\ \mathbf{r}(u_k, v_k+\Delta v) - \mathbf{r}(u_k, v_k) \simeq t_v \Delta v \end{array}$

$\therefore$ The area of the flat parallelogram under the curved patch is equal to the magnitud of the cross product of the vectors $|(t_u \Delta u \times t_v \Delta v)|$

$$\therefore \Delta S \simeq |t_u \times t_v| \, \Delta u \, \Delta v$$

$$= |t_u \times t_v| \, \Delta A$$



Now, lets compute n, the normal vector

The required vector normal to the surface at a point is: $\dfrac{t_u \times t_v}{|t_u \times t_v|}$

Surface Integral of a Vector Field

Def. Suppose $F = \langle f, g, h \rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth parametrically oriented surface S. If S is defined as $r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, for (u,v) in a region R,

$$\iint_S F \cdot n \, dS = \iint_R F \cdot \underbrace{\frac{t_u \times t_v}{\cancel{|t_u \times t_v|}}}_{n} \underbrace{\cancel{|t_u \times t_v|} \, dA}_{dS}$$

$$= \iint_R F \cdot (t_u \times t_v) \, dA .$$

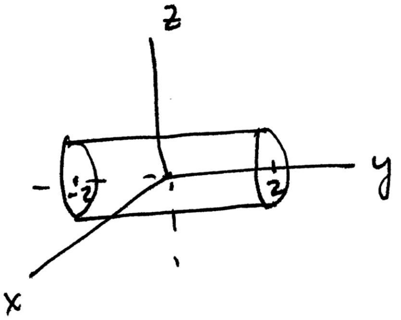
where $t_u = \dfrac{\partial r}{\partial u}$, and $t_v = \dfrac{\partial r}{\partial v}$ are continuous on R

the normal vector $t_u \times t_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S.

Ex Find the flux of the vector field

$$F = \frac{\langle x, 0, z \rangle}{\sqrt{x^2+z^2}}$$ across the surface S: $x^2+z^2=a^2$, $|y| \le 2$

1. the surface S is a cylinder



Need to parameterize S:

$x = a\cos u$

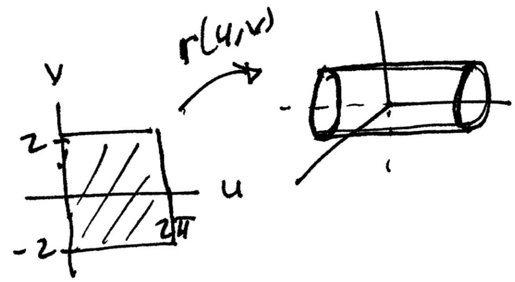
$y = v$

$z = a\sin u$

$0 \le u \le 2\pi$

$-2 \le v \le 2$

$\therefore\ S: \mathbf{r}(u,v) = \langle a\cos u, v, a\sin u \rangle.$



2. need to compute $t_u \times t_v$.

$t_u = \frac{\partial \mathbf{r}}{\partial u} = \langle -a\sin u, 0, a\cos u \rangle$

$t_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 0, 1, 0 \rangle$

$$t_u \times t_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u & 0 & a\cos u \\ 0 & 1 & 0 \end{vmatrix} = \langle -a\cos u, 0, -a\sin u \rangle$$

we want $t_u \times t_v$ to point outward, then we need to change the sign:

$t_u \times t_v = \langle a\cos u, 0, a\sin u \rangle$

3. Evaluating F at the points on the surface we get:

$$F = \frac{\langle a\cos u,\ 0,\ a\sin u \rangle}{\sqrt{a^2\cos^2 u + a^2\sin^2 u}} = \frac{\langle a\cos u,\ 0,\ a\sin u\rangle}{a}$$

$$= \langle \cos u,\ 0,\ \sin u \rangle$$

4. The integrand then is:

$$F \cdot (t_u \times t_v) = \langle \cos u,\ 0,\ \sin u\rangle \cdot \langle a\cos u,\ 0,\ a\sin u\rangle$$

$$= a$$

$$\therefore \iint_S F \cdot n\, dS = \iint_R a\ dA$$

$$= \int_{-2}^{2} \int_{0}^{2\pi} a\ du\, dv$$

$$= \underline{8\pi a}$$

Def **Surface Integral of a scalar field.**

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, for (u,v) in a region R,

The surface integral of f over S is:

$$\iint_S f \, dS = \iint_R f \underbrace{|t_u \times t_v| \, dA}_{dS}$$

where $t_u = \dfrac{\partial \mathbf{r}}{\partial u}$, and $t_v = \dfrac{\partial \mathbf{r}}{\partial v}$ are cont on R.

The normal vector $t_u \times t_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S.