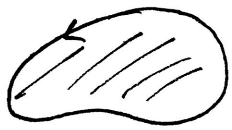
15.7 Stoke's Theorem

Green's Thm
circulation form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$



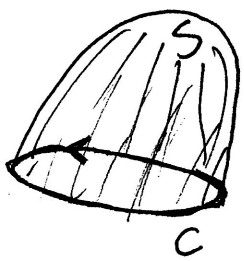
$$= \iint_R (\nabla \times \mathbf{F}) \cdot \hat{k} \; dA$$

Stokes Thm

Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous partial derivatives on S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \; dS$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.



Ex Verifying Stokes' Thm.

Confirm that Stokes' theorem holds for the vector field $F = \langle z-y,\ x,\ -x \rangle$, where S is the semisphere $x^2+y^2+z^2=4$, for $z \geq 0$, and C is the circle $x^2+y^2=4$ oriented counterclockwise

First lets compute $\oint_C F \cdot dr$ from definition:

- to compute this line integral, first we parameterize the curve C, (counterclockwise).

$C: \ r(t) = \langle 2\cos t,\ 2\sin t,\ 0 \rangle \qquad 0 \leq t \leq 2\pi$

- next, we evaluate the vector field at all points of C

$F = \langle 0 - 2\sin t,\ 2\cos t,\ -2\cos t \rangle$

- now, need to compute $r'(t)$:

$r'(t) = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$

$\therefore$ The integrand of the line integral is:

$F \cdot r'(t) = 4\sin^2 t + 4\cos^2 t = 4$

$\therefore \oint_C F \cdot dr = \int_0^{2\pi} 4\, dt = \underline{\underline{8\pi}}$

Now, lets use Stokes' thm.

Need to evaluate $\iint_S (\nabla \times F) \cdot \mathbf{n}\, dS$ and

confirm that the value is 8π,

$$\iint_S (\nabla \times F) \cdot \mathbf{n}\, dS = \iint_R (\nabla \times F) \cdot (t_u \times t_v)\, dA$$

- need to parameterize S

$S: x^2+y^2+z^2 = 4, \quad z \geq 0$

$z = \sqrt{4-x^2-y^2}$

$\therefore S: \mathbf{r}(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle$

$R: x^2+y^2 \leq 4$

(R is the region where x and y take their values to describe the surface, for an explicitly defined surface, R is the domain (or shadow) of the function $z = f(x,y)$).

- now, lets go after the task of computing $(t_x \times t_y)$

$t_x = \dfrac{\partial \mathbf{r}}{\partial x} = \left\langle 1, 0, \dfrac{-x}{\sqrt{4-x^2-y^2}} \right\rangle$

$t_y = \dfrac{\partial \mathbf{r}}{\partial y} = \left\langle 0, 1, \dfrac{-y}{\sqrt{4-x^2-y^2}} \right\rangle$

$$t_x \times t_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{-x}{\sqrt{4-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{4-x^2-y^2}} \end{vmatrix}$$

$$= \left\langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 \right\rangle$$

- next, lets compute $\nabla \times F$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z-y & x & -x \end{vmatrix} = \langle 0, 2, 2 \rangle$$

$\therefore$ The integrand is:

$$(\nabla \times F) \cdot (t_x \times t_y) = \langle 0,2,2 \rangle \cdot \left\langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 \right\rangle$$

$$= \frac{2y}{\sqrt{4-x^2-y^2}} + 2$$

$$\therefore \iint_R (\nabla \times F) \cdot (t_x \times t_y)\, dA = \iint_R \left(\frac{2y}{\sqrt{4-x^2-y^2}} + 2 \right) dA$$

the double integral on the right hand side could be best computed using polar coordinates:

$$= \int_0^{2\pi} \int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4-r^2}} + 2 \right) r\, dr\, d\theta = \dots = \underline{8\pi}$$

Example continues...

Stokes' theorem says that the line integral over the closed curve C is equal to a surface integral over a surface that has C as its boundary, this S could be any surface, as long as the boundary is C.

Lets compute the surface integral in this example by changing the surface S.

Instead of taking the semisphere $x^2+y^2+z^2=4,\ z\geq 0$

We could take the disk $x^2+y^2 \leq 4$

notice that the boundary of this disk is indeed C.

- $S: \ x^2+y^2 \leq 4$

parameterize S: $r(u,v) = \langle v\cos u,\ v\sin u,\ 0\rangle$

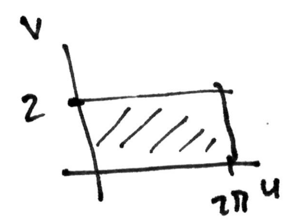
$0 \leq u \leq 2\pi$

$0 \leq v \leq 2$

- $t_u = \dfrac{\partial r}{\partial u} = \langle -v\sin u,\ v\cos u,\ 0\rangle$

$t_v = \dfrac{\partial r}{\partial v} = \langle \cos u,\ \sin u,\ 0\rangle$

R:



$t_u \times t_v = \langle 0,\ 0,\ -v\rangle$

... need to change the sign, to keep the normal pointing upwards: $\langle 0, 0, v\rangle$

$\cdot \ \nabla \times F = \langle 0, 2, 2 \rangle$ (found before)

$\therefore$ The integrand is:

$$(\nabla \times F) \cdot (t_u \times t_v) = \langle 0,2,2 \rangle \cdot \langle 0,0,v \rangle = 2v$$

$$\therefore \iint_R (\nabla \times F) \cdot (t_u \times t_v)\, dA = \iint_R 2v \, dA$$

$$= \int_0^2 \int_0^{2\pi} (2v)\, du\, dv$$

$$= \int_0^2 2vu \Big|_0^{2\pi} dv$$

$$= \int_0^2 4\pi v \, dv$$

$$= 2\pi v^2 \Big|_0^2 = \underline{8\pi}$$