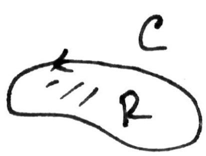
15.8 Divergence Thm.

Flux calculations may be done using flux integrals.
The divergence theorem offers an alternative method. It says that instead of integrating the flow in and out of a region across its boundary, you may also add up all the sources (or sinks) of the flow throughout the region.



Flux form of Green's Thm

$$\oint_C F \cdot n\, ds = \iint_R (\nabla \cdot F)\, dA$$

Divergence Thm

$$\iint_S F \cdot n\, dS = \iiint_D (\nabla \cdot F)\, dV$$

<u>Thm</u> <u>Divergence</u>

Let F be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

$$\iint_S F \cdot n\, dS = \iiint_D (\nabla \cdot F)\, dV$$

where n is the outward unit normal vector on S.

Ex Compute the net outward flux of

$F = \langle y - 2x, \quad x^3 - y, \quad y^2 - z \rangle$

S is the sphere $x^2 + y^2 + z^2 = 4$.

$$\nabla \cdot F = (-2) + (-1) + (-1) = -4$$

By divergence Thm:

$$\text{flux} = \iint_S F \cdot n \, dS = \iiint_D \nabla \cdot F \, dV$$

$$= \iiint_D (-4) \, dV$$

$$= (-4) \text{ volume}(D)$$

$$= -4 \left(\tfrac{4}{3} \pi (2)^3 \right)$$

$$= \frac{-128\pi}{3}.$$

Ex Heat flux

The heat flow vector field for conducting objects is $F = -k\nabla T$, where $T(x,y,z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of F across the surface S for the given temperature distributions. Assume $k = 1$.

$$T(x,y,z) = -\ln(x^2+y^2+z^2)$$

S is the sphere $x^2+y^2+z^2 = 4$

$$F = -k\nabla T = -\nabla T = \left\langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right\rangle$$

$$\text{flux} = \iint_S F \cdot n \, dS \underset{\substack{\uparrow \\ \text{div thm}}}{=} \iiint_D (\nabla \cdot F)\, dV$$

$$\nabla \cdot F = \frac{-\cancel{2x^2}+\cancel{2y^2}+\cancel{2z^2}}{(x^2+y^2+z^2)^2} + \frac{\cancel{2x^2}-2y^2+2z^2}{(x^2+y^2+z^2)^2} + \frac{2x^2+2y^2\cancel{-2z^2}}{(x^2+y^2+z^2)^2}$$

$$= \frac{2\,\cancel{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^{\cancel{2}}} = \frac{2}{x^2+y^2+z^2}$$

$$\therefore \iiint_D \nabla \cdot F \, dV = \iiint_D \frac{2}{(x^2+y^2+z^2)}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \frac{2}{\cancel{\rho^2}} \cancel{\rho^2} \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \dots = 16\pi$$