Briggs/Cochran/Gillett
Calculus for Scientists and Engineers
Multivariable Study Card

Vectors and the Geometry of Space

Three-Dimensional Coordinate Systems

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

The standard equation for the sphere of radius r and center (a, b, c) is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Vectors

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be vectors and c a scalar.

The magnitude or length of a vector $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

The vector with initial point $P = (x_1, y_1, z_1)$ and terminal point $Q = (x_2, y_2, z_2)$ is $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Operations on Vectors

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$

Dot Product: $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$. Because $\cos\frac{\pi}{2} = 0$, we have that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal or perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Also, $\mathbf{u} \cdot \mathbf{u} = |u|^2$. The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$.

Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Cross Product Properties

1. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta =$ area of parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
2. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
3. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
4. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, therefore $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.
5. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$ volume of parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Vector-valued Functions and Motion in Space

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function. Then $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$ is the velocity vector and $|\mathbf{v}(t)|$ is the speed. The acceleration vector is $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$. The unit tangent vector is $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ and the length of $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$L = \int_a^b |\mathbf{v}|\,dt.$$

Lines in Space

A vector equation of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$,

or $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$, for $-\infty < t < \infty$.

Parametric equations for this line are $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, for $-\infty < t < \infty$.

Formulas

Curvature: $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Radius of Curvature: $\rho = \frac{1}{\kappa}$

Principal Unit Normal: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Tangential and normal scalar components of acceleration:

$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where

$a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|}$ and $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|}$.

Unit Binormal Vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Torsion: $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}$

Planes and Surfaces in Space

Planes

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has vector equation:

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

component equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

component equation simplified:

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$

Quadric Surfaces

1. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Elliptic Paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
3. Elliptic Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
4. Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
5. Hyperboloid of two sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
6. Hyperbolic Paraboloid: $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Partial Derivatives

To compute $\frac{\partial f}{\partial x}$, differentiate $f(x, y)$ treating y as a constant.

To compute $\frac{\partial f}{\partial y}$, differentiate $f(x, y)$ treating x as a constant.

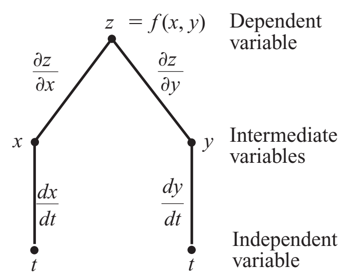
Thus, if $f(x, y) = y \cos xy$, $\frac{\partial f}{\partial x} = -y^2 \sin xy$, and $\frac{\partial f}{\partial y} = \cos xy - xy \sin xy$.

Notation

$$\frac{\partial f}{\partial x} = f_x, \frac{\partial f}{\partial y} = f_y$$

Chain Rule

To find dz/dt, start at z and read down each branch to t, multiplying derivatives along the way. Then add the products.



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Gradient Vector

The gradient vector (gradient) of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Directional Derivative

The directional derivative of f at (a, b) in the direction of unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Plane Tangent to a Surface

The tangent plane at the point $P_0(a, b, c)$ on the level surface $F(x, y, z) = 0$ is the plane through P_0 normal to $\nabla F\,|_{P_0}$.

Tangent Plane for $F(x, y, z) = 0$ **at** $P_0(a, b, c)$

$$F_x(P_0)(x - a) + F_y(P_0)(y - b) + F_z(P_0)(z - c) = 0$$

Plane Tangent to a Surface $z = f(x, y)$

The plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Linearization

The linearization of a function $f(x, y)$ at a point (a, b) where f is differentiable is

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Second Derivative Test for Local Extrema

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- **i.** f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **ii.** f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **iii.** f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
- **iv.** **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

Partial Derivatives (continued)

Lagrange Multipliers

One Constraint: Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

Two Constraints: For constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, g and h differentiable, find the values of x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g(x, y, z) = 0, \quad \text{and} \quad h(x, y, z) = 0.$$

Multiple Integrals

Double Integrals as Volumes

When $f(x, y)$ is a positive function over a region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$.

This volume can be evaluated by computing an iterated integral.

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous of $[c, d]$, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

Area via Double Integral

$$A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx$$

Area in Polar Coordinates

$$A = \iint_R r\,dr\,d\theta$$

Triple Integrals

$$\iiint_D f(x, y, z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x, y)}^{H(x, y)} f(x, y, z)\,dz\,dy\,dx,$$

where $D = \{(x, y, z): a \le x \le b, g(x) \le y \le h(x), G(x, y) \le z \le H(x, y)\}$.

Cylindrical Coordinates (r, θ, z)

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r^2 = x^2 + y^2$

$\tan\theta = y/x$

Triple Integrals in Cylindrical Coordinates

$$\iiint_D f(r, \theta, z)\,dV = \iiint_D f(r, \theta, z)\,dz\,r\,dr\,d\theta$$

Spherical Coordinates (ρ, φ, θ)

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$x = r\cos\theta = \rho\sin\varphi\cos\theta$

$y = r\sin\theta = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$r = \rho\sin\varphi \qquad \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$

Triple Integrals in Spherical Coordinates

$$\iiint_D f(\rho, \varphi, \theta)\,dV = \iiint_D f(\rho, \varphi, \theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Change of Variables Formula for Double Integrals

$$\iint_R f(x, y)\,dy\,dx = \iint_S f(g(u, v), h(u, v))\,|J(u, v)|\,dA,$$

where $x = g(u, v)$, $y = h(u, v)$ take region S onto region R and

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

is the Jacobian determinant.

Integration in Vector Fields

Line Integrals

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parameterization of C,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\,|\mathbf{v}(t)|\,dt,$$

where $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$ and $ds = |\mathbf{v}(t)|\,dt$.

Work

The work done by a force $\langle f, g, h\rangle$ over a smooth curve $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\,dt = \int_a^b \mathbf{F} \cdot d\mathbf{r} =$$

$$\int_C f\,dx + g\,dy + h\,dz.$$

Circulation

The circulation of $\mathbf{F}$ on C is $\oint_C \mathbf{F} \cdot \mathbf{T}\,ds$.

Flux

Flux of $\mathbf{F}$ across $C = \oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx$ where $\mathbf{F} = \langle f, g\rangle$ and $\mathbf{n}$ is outward pointing normal along C.

Conservative Vector Field

$\mathbf{F}$ is conservative if $\mathbf{F} = \nabla\varphi$ for some function $\varphi(x, y, z)$. If $\mathbf{F}$ is conservative, then

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

is independent of path and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A).$$

Also,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

around every closed curved C in this case.

Component Test for Conservative Fields

Let $\mathbf{F} = \langle f, g, h\rangle$ be a field whose component functions have continuous first partial derivatives on an open, simply connected region. Then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

Green's Theorem

If C is a simple closed curve and R is the region enclosed by C then

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\,ds}_{\text{Outward flux}} = \oint_C f\,dy - g\,dx = \underbrace{\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA}_{\text{Divergence integral}}$$

and

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\,ds}_{\text{Counterclockwise circulation}} = \oint_C f\,dx + g\,dy = \underbrace{\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.}_{\text{Curl integral}}$$

Green's Theorem Area Formula

$$\text{Area of } R = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

Surface Integrals

Let $\mathbf{r}(u, v) = \langle f(u, v), y(u, v), z(u, v)\rangle$ $a \le u \le b, c \le v \le d$ be a parameterization of a surface S.

Let $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v}\right\rangle$.

The unit vector normal to the surface is

$$\frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|} = \mathbf{n}.$$

Area

The area of the surface S is $\displaystyle\int_c^d \int_a^b |\mathbf{t}_u \times \mathbf{t}_v|\,du\,dv$.

Integration in Vector Fields (continued)

Surface Integral

If $f(x, y, z)$ is defined over S then the integral of f over S is

$$\iint_S f(x, y, z)\, dS =$$

$$\int_c^d \int_a^b f(x(u, v), y(u, v), z(u, v))\left|\mathbf{t}_u \times \mathbf{t}_v\right| du\, dv.$$

Curl of a Vector Field

If $\mathbf{F} = \langle f, g, h \rangle$, then

$$\text{curl } \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

$$= \nabla \times \mathbf{F}.$$

Divergence of a Vector Field

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Stokes' Theorem

The circulation of a vector field $\mathbf{F} = \langle f, g, h \rangle$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS}_{\text{Curl integral}}$$

Divergence Theorem

The flux of a vector field $\mathbf{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, dS}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{Divergence integral}}.$$

Green's Theorem and Its Generalization to Three Dimensions

Circulation form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

Flux form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$