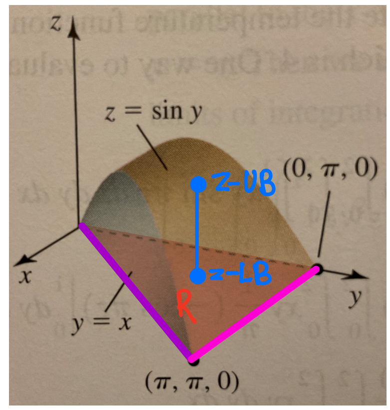
Midterm 3 Review

You should attempt these questions using only the equation sheet of midterm 3. No calculators or other resources allowed.

1) Set up a triple integral to find the volume of the following solid in the first octant, bounded by the cylinder $z = \sin y$, and sliced by the planes $y = x$ and $x = 0$.

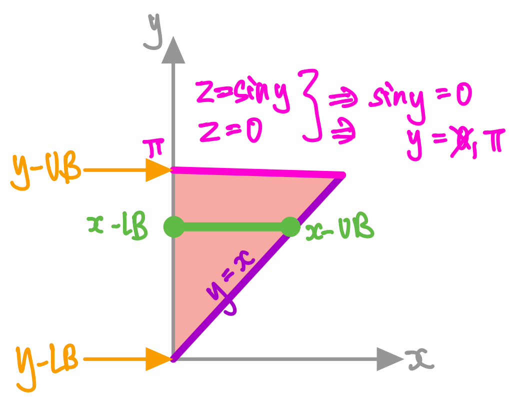


Hint: Integrate in the order $dz\,dx\,dy$.

z- LB: $z = 0$
UB: $z = \sin y$

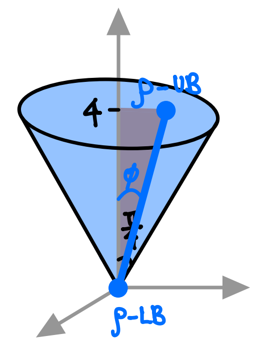
x- LB: $x = 0$
UB: $x = y$

y- LB: $y = 0$
UB: $y = \pi$

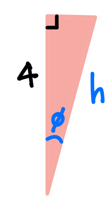


$$\therefore \boxed{\text{Vol} = \int_0^{\pi} \int_0^{y} \int_0^{\sin y} 1 \; dz\,dx\,dy}$$

2) Set up a triple integral in spherical coordinates to calculate the volume of the cone to the right, bounded by the equations $\phi = \frac{\pi}{4}$ and $z = 4$.



Hint: Integrate in the order $d\rho\, d\phi\, d\theta$.



$\cos\phi = \frac{4}{h}$

$\Rightarrow h = 4\sec\phi$

ρ - LB: $\rho = 0$

UB: $\rho = h = 4\sec\phi$

ϕ - LB: $\phi = 0$

UB: $\phi = \frac{\pi}{4}$

θ - LB: $\theta = 0$

UB: $\theta = 2\pi$

$$\therefore \text{Vol} = \iiint 1 \, dV$$

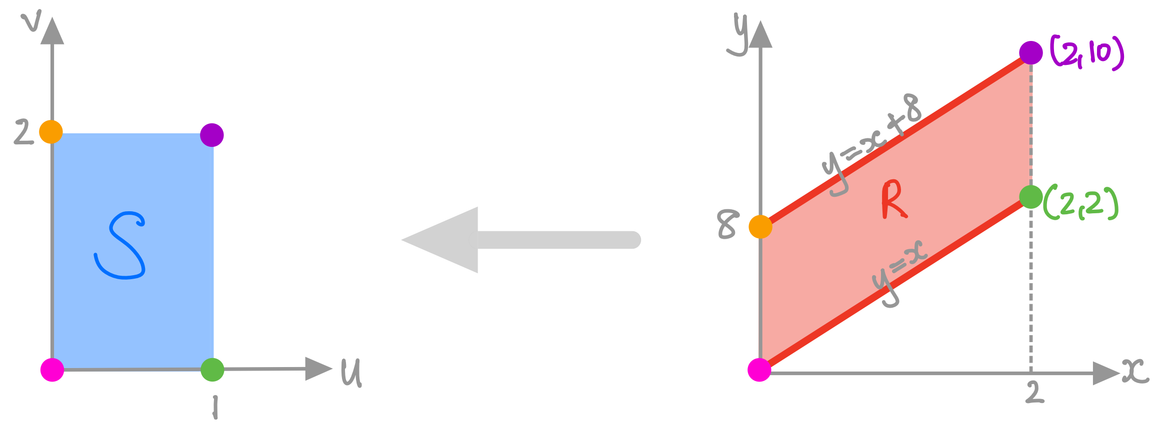
$$= \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \boxed{\int_0^{2\pi} \int_0^{\pi} \int_0^{4\sec\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta}$$

3) Set up a double integral to evaluate the following integral

$$\iint_R x^2 y \, dA, \quad R = \left\{ (x,y) : \begin{array}{l} 0 \le x \le 2 \\ x \le y \le x+8 \end{array} \right\}$$

using change of variables $\begin{cases} x = 2u \\ y = 4v + 2u. \end{cases}$



$$\left.\begin{array}{l} x = 2u \\ y = 4v + 2u \end{array}\right\} \Rightarrow \begin{array}{l} u = \frac{1}{2}x, \\ v = \frac{1}{4}(y - 2u) = \frac{1}{4}(y - x) \end{array}$$

$$\left.\begin{array}{l} (0,0) \leftarrow (0,0) \\ (1,0) \leftarrow (2,2) \\ (1,2) \leftarrow (2,10) \\ (0,2) \leftarrow (0,8) \end{array}\right\} \quad S = \left\{ (u,v) : \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 2 \end{array} \right\}$$

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & 4 \end{vmatrix} = 8 - 0 = 8$$

$$\iint_R x^2 y \, dA = \iint_S x^2 y \, |J(u,v)| \, du \, dv = \boxed{\int_0^2 \int_0^1 (2u)^2 (4v+2u) \cdot |8| \, du \, dv}$$

4) Let $\mathcal{C}$ be the oriented parabola $\vec{r}(t) = \langle 4t, t^2 \rangle$ for $0 \le t \le 1$.

Let $\vec{F} = \langle xy, y \rangle$ be a vector field across $\mathcal{C}$. Compute:

a) the circulation of $\vec{F}$ on $\mathcal{C}$

b) — flux ————————.

a) Circulation $= \int_{\mathcal{C}} \vec{F} \cdot \vec{r}\,' \, dt$

$$= \int_0^1 \langle 4t^3, t^2 \rangle \cdot \langle 4, 2t \rangle \, dt$$

$$= \int_0^1 16t^3 + 2t^3 \, dt = \cdots = \boxed{\frac{9}{2}}$$

b) Flux $= \int_{\mathcal{C}} \vec{F} \cdot \langle y', -x' \rangle \, dt$

$$= \int_0^1 \langle 4t^3, t^2 \rangle \cdot \langle 2t, -4 \rangle \, dt$$

$$= \int_0^1 8t^4 - 4t^2 \, dt = \cdots = \boxed{\frac{4}{15}}$$

5) Is $\vec{F} = \langle z, 1, x \rangle$ conservative? If so, find a potential function.

$$\begin{matrix} f & g & h \end{matrix}$$

$$\left.\begin{array}{l} f_y = 0 = g_x \\ f_z = 1 = h_x \\ g_z = 0 = h_y \end{array}\right\} \Rightarrow \boxed{\text{Conservative.}}$$

$$\vec{F} = \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle z, 1, x \rangle$$

$$\varphi(x,y,z) = \int z\,dx = zx + G(y,z)$$

$$\Rightarrow \varphi_y = 0 + G_y = 1$$

$$\Rightarrow \quad G_y(y,z) = \int 1\,dy = y + H(z)$$

$$\Rightarrow \varphi = zx + y + H(z)$$

$$\Rightarrow \varphi_z = \cancel{x} + 0 + H'(z) = \cancel{x}$$

$$\Rightarrow \quad H'(z) = 0$$

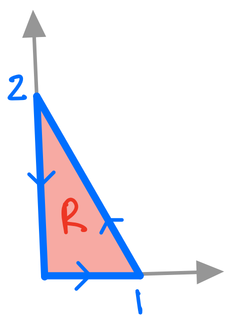
$$\Rightarrow \quad H(z) = \int 0\,dz = k \overset{\text{set}}{=} 0$$

$$\therefore \varphi(x,y,z) = zx + y + 0 = \boxed{zx + y}$$

6) Compute $\oint_{\mathcal{C}} \overbrace{-3y}^{f}\, dy - \overbrace{3x}^{g}\, dx$, where $\mathcal{C}$ is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$, oriented counterclockwise.

Apply Green's circulation form with

$$f = -3y$$
$$g = 3x.$$



$$\text{Circulation} = \iint_R g_x - f_y \, dA$$

$$= \iint_R 3+3 \, dA$$

$$= 6 \underbrace{\iint_R 1 \, dA}_{= \text{Area of } R}$$

$$= 6 \times \left(\tfrac{1}{2} \times 1 \times 2\right) = \boxed{6}$$

7) Let $\vec{F} = \langle \overset{f}{x\cos y}, \overset{g}{y^2}, \overset{h}{xz} \rangle$. Compute:

a) div $\vec{F}$

b) curl $\vec{F}$

c) the divergence of curl $\vec{F}$

a) $\nabla \cdot \vec{F} = f_x + g_y + h_z$

$= \boxed{\cos y + 2y + x}$

b) $\nabla \times \vec{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$

$= \langle 0 - 0, 0 - z, 0 + x\sin y \rangle$

$= \boxed{\langle 0, -z, x\sin y \rangle}$.

c) $\nabla \cdot (\nabla \times \vec{F})$ is always $\boxed{0}$.