$\boxed{\text{Review (Finals)}}$

17.8 : Use the divergence theorem to compute net outward flux of $F = \langle x, -2y, 3z \rangle$ where S is the sphere $x^2+y^2+z^2=6$.

Ans:

$$\iint_S F \cdot n \, dS = \iiint_D \nabla \cdot F \, dV$$

$$\nabla \cdot F = 1 + -2 + 3 = 2$$

$$= \iiint_D 2 \, dV = 2 \cdot \text{volume of sphere of radius } \sqrt{6}$$

$$= 2 \cdot \frac{4}{3} \pi (\sqrt{6})^3$$

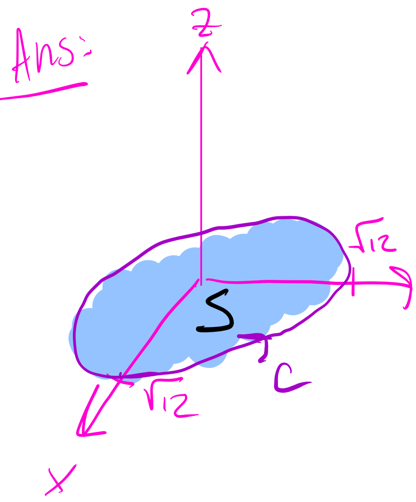
$$= \frac{2 \cdot 4 \cdot 6 \cdot \sqrt{6} \cdot \pi}{3}$$

$$= \boxed{16\sqrt{6}\pi}$$

17.7: Compute the line integral $\oint F \cdot dr$ by setting up the surface integral in Stokes' Theorem with an appropriate choice of S. Assume C has counterclockwise orientation.

$F = \langle y, -z, x \rangle$; C is the circle $x^2+y^2=12$ in the plane $z=0$.

Ans:



$S = r(u,v) = \langle \overset{x}{u\cos v}, \overset{y}{u\sin v}, \overset{z}{0} \rangle$ $\quad 0 \le u \le \sqrt{12}$, $\quad 0 \le v \le 2\pi$

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dS = \iint_R \underline{(\nabla \times F)} \cdot \underline{(t_u \times t_v)} \, dA$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -z & x \end{vmatrix} = (0-(-1))i - (1-0)j + (0-1)k$$

$$= \langle 1, -1, -1 \rangle = \langle 1, -1, -1 \rangle$$

$t_u = \langle \cos v, \sin v, 0 \rangle$

$t_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0, 0, u \rangle \checkmark$$

$$\boxed{\int_{u=0}^{u=\sqrt{12}} \int_{v=0}^{v=2\pi} -u \, dv \, du}$$

17.3: Is $\langle \overset{f}{yz}, \overset{g}{xz}, \overset{h}{xy} \rangle$ conservative?

Ans:

$$\left.\begin{aligned} f_y &= z = g_x \\ f_z &= y = h_x \\ g_z &= x = h_y \end{aligned}\right\} \text{yes!}$$

17.2: Evaluate $\int_C F\cdot T\,ds$ where

$F = \langle -y, x\rangle$ on the parabola $y = x^2$

from $(0,0)$ to $(1,1)$.

Answer: $r(t) = \langle \overset{x}{t}, \overset{y}{t^2}\rangle$ $\quad r'(t) = \langle 1, 2t\rangle$

$0 \le t \le 1$

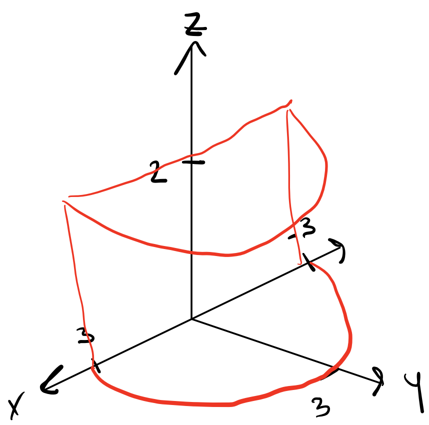
$$\int_C F\cdot T\,ds = \int_C F\cdot r'(t)\,dt = \int_{t=0}^{t=1} \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle\,dt$$

$$= \int_{t=0}^{t=1} -t^2 + 2t^2\,dt = \int_{t=0}^{t=1} t^2\,dt = \frac{t^3}{3}\Big|_{t=0}^{t=1}$$

$$= \boxed{\frac{1}{3}}$$

16.5. Convert the following to cylindrical coordinates:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2} \frac{1}{1+x^2+y^2}\, dz\, dy\, dx$$



Ans:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

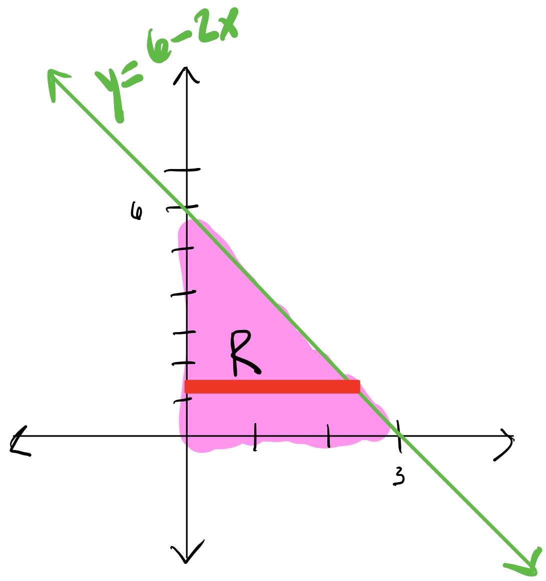
$dV = r\, dz\, dr\, d\theta$

$x^2+y^2 = r^2$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}\int_{z=0}^{z=2} \frac{1}{1+r^2}\, r\, dz\, dr\, d\theta$$

16.2: Reverse the order of integration in the following integral.

$$\int_0^3 \int_0^{6-2x} f(x,y)\,dy\,dx$$



Answer:

$$\int_{x=0}^{x=3} \int_{y=0}^{y=6-2x} f(x,y)\,dy\,dx = \int_{y=0}^{y=6} \int_{x=0}^{x=-\frac{1}{2}y+3} f(x,y)\,dx\,dy$$

15.8 Find the absolute maximum value of $f(x,y) = x+2y$ subject to the constraint $x^2+y^2=4$.

Answer: $g(x,y) = x^2+y^2-4 = 0$

$$\nabla f = \lambda \nabla g$$
$$\nabla f = \langle 1, 2 \rangle$$
$$\nabla g = \langle 2x, 2y \rangle$$

$$\langle 1, 2 \rangle = \lambda \langle 2x, 2y \rangle$$

1. $1 = \lambda 2x \Rightarrow \lambda = \frac{1}{2x}$
2. $2 = \lambda 2y \qquad \lambda = \frac{1}{y}$
3. $x^2+y^2-4 = 0$

$\frac{1}{2x} = \frac{1}{y}$ $\quad \boxed{2x = y}$

$x \neq 0$
$y \neq 0$

$$x^2 + (2x)^2 - 4 = 0$$
$$x^2 + 4x^2 - 4 = 0$$
$$5x^2 - 4 = 0$$
$$5x^2 = 4$$
$$x^2 = \frac{4}{5}$$
$$x = \pm \frac{2}{\sqrt{5}}$$

Table:

x	y	λ	$f(x,y) = x+2y$
$\frac{2}{\sqrt{5}}$	$\frac{4}{\sqrt{5}}$	$\frac{\sqrt{5}}{4}$	$\frac{10}{\sqrt{5}}$
$-\frac{2}{\sqrt{5}}$	$\frac{-4}{\sqrt{5}}$	$\frac{\sqrt{5}}{-4}$	$\frac{-10}{\sqrt{5}}$

Absolute max: $\frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$

15.7 Find all critical point(s) of the function $f(x,y) = \frac{x^3}{3} - \frac{y^3}{3} + 3xy$.

For each point(s), determine if it is a local min/max/saddle point.

$$\left.\begin{aligned} f_x &= x^2 + 3y \overset{\text{set}}{=} 0 \\ f_y &= -y^2 + 3x \overset{\text{set}}{=} 0 \end{aligned}\right\} \Rightarrow \quad y = -\frac{1}{3}x^2$$

$$-\left(\frac{1}{3}x^2\right)^2 + 3x = 0$$

$$\Rightarrow 3x = \frac{x^4}{9}$$

$$\Rightarrow x^4 - 27x = 0$$

$$\Rightarrow x(x^3 - 27) = 0$$

$$\Rightarrow x = 0 \text{ or } x^3 = 27$$

$$\Rightarrow x = 0 \text{ or } x = 3.$$

Case $x=0$: $y = -\frac{1}{3} \cdot 0^2 = 0$

Case $x=3$: $y = -\frac{1}{3} \cdot 3^2 = -3$

$\therefore$ Critical points are $\boxed{(0,0),\ (3,-3)}$.

$f_{xx} = 2x$, $f_{yy} = -2y$, $f_{xy} = 3$

$\therefore D = f_{xx} f_{yy} - f_{xy}^2 = -4xy - 9.$

$D(0,0) = -0-9 = -9 < 0 \Rightarrow \boxed{(0,0) \text{ is saddle point.}}$

$D(3,-3) = -4 \cdot 3 \cdot (-3) - 9 > 0$, $f_{xx}(3,-3) = 2 \cdot 3 > 0$

$\Rightarrow \boxed{(3,-3) \text{ is local min.}}$

15.6. Find an equation of the plane that is tangent to the surface $\dfrac{x+z}{y-z^3} = 2$ at the point $P(2,1,0)$.

$$x+z = 2y-2z^3$$

$$\underbrace{x-2y+2z^3+z}_{F(x,y,z)} = 0$$

$$\nabla F = \langle F_x, F_y, F_z \rangle$$

$$= \langle 1, -2, 6z^2+1 \rangle$$

$$\nabla F(2,1,0) = \langle 1,-2,1 \rangle$$

Plane $\perp \langle 1,-2,1 \rangle$, pass through $\langle 2,1,0 \rangle$

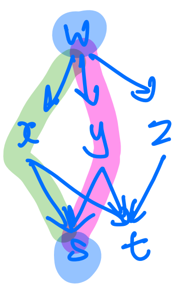
$\Rightarrow$ Eqn : $\langle 1,-2,1 \rangle \cdot \langle x,y,z \rangle = \langle 1,-2,1 \rangle \cdot \langle 2,1,0 \rangle$

$$x-2y+z = 2-2+0 = 0$$

$$\boxed{x-2y+z = 0}$$

15.4. Find w_s (in terms of s & t), if

$$w = x^2y + xz + ye^z,$$
$$x = s+t,$$
$$y = st,$$
$$z = t^3.$$



$$w_s = w_x \cdot x_s + w_y \cdot y_s$$
$$= (2xy+z)\cdot 1 + (x^2+e^z)\cdot t$$
$$= 2xy + z + x^2t + e^zt.$$
$$= \boxed{2(s+t)(st) + t^3 + (s+t)^2t + e^{t^3}t}.$$

15.2. Use the two-path test to show that the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^3+y^2}$$

does not exist.

Limit along x-axis $(y=0)$: $\displaystyle\lim_{x\to 0} \frac{x^3-0^2}{x^3+0^2} = \lim_{x\to 0} \frac{x^3}{x^3} = \lim_{x\to 0} 1 = 1$

Limit along y-axis $(x=0)$: $\displaystyle\lim_{y\to 0} \frac{0-y^2}{0+y^2} = \lim_{y\to 0} -\frac{y^2}{y^2} = \lim_{y\to 0} -1 = -1$

Limits along axes different $\Rightarrow$ Limit does not exist. $\square$

14.5. Find the curvature $\kappa(t)$ of the curve $\vec{r}(t) = \langle \sqrt{3}\sin t, \sin t, 2\cos t \rangle$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle \sqrt{3}\cos t, \cos t, -2\sin t \rangle$$

$$|\vec{v}(t)| = \sqrt{3\cos^2 t + \cos^2 t + 4\sin^2 t}$$

$$= \sqrt{4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{4} = 2$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{2}\langle \sqrt{3}\cos t, \cos t, -2\sin t \rangle$$

$$\frac{d\vec{T}}{dt} = \frac{1}{2}\langle -\sqrt{3}\sin t, -\sin t, -2\cos t \rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{2}\sqrt{3\sin^2 t + \sin^2 t + 4\cos^2 t}$$

$$= \frac{1}{2}\sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \frac{1}{2}\sqrt{4} = \frac{1}{2}\times 2 = 1$$

$$\kappa(t) = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{2}\cdot 1 = \boxed{\frac{1}{2}}$$

14.4. Find the arc length of the curve $\vec{r}(t) = \langle 2t^3, -t^3, t^3 \rangle$, $0 \le t \le 1$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 6t^2, -3t^2, 3t^2 \rangle$$

$$|\vec{v}(t)| = \sqrt{36t^4 + 9t^4 + 9t^4} = \sqrt{54}\,t^2$$

Length $s(1) = \int_0^1 \sqrt{54}\,u^2\,du$

$$= \frac{\sqrt{54}}{3} u^3 \Big|_0^1$$

$$= \boxed{\frac{\sqrt{54}}{3}} = \frac{\sqrt{6 \times 3^2}}{3} = \frac{3\sqrt{6}}{3} = \sqrt{6}$$

13.5.1 Find an equation of the line that passes through

$\langle 1,1,1\rangle$ and $\langle 2,3,-2\rangle$.

Line $\parallel$ $\langle 2-1, 3-1, -2-1\rangle = \langle 1,2,-3\rangle$

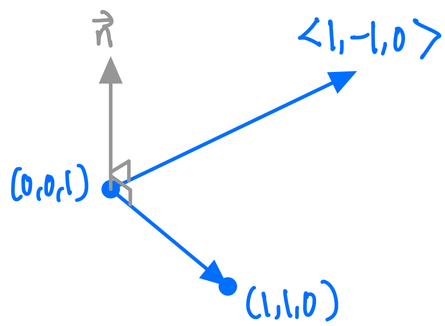
$$\boxed{\vec{r}(t) = \langle 1,2,-3\rangle t + \langle 1,1,1\rangle}.$$

13.5.2. Find an equation of the plane that contains the point

$P(1,1,0)$ and the line $\vec{r}(t) = \langle 1,-1,0\rangle t + \langle 0,0,1\rangle$.

Plane $\parallel$ $\langle 1,-1,0\rangle$,

$\langle 1,1,0\rangle - \langle 0,0,1\rangle = \langle 1,1,-1\rangle$



$\Rightarrow \vec{n} \perp \langle 1,-1,0\rangle, \langle 1,1,-1\rangle$

$\Rightarrow \vec{n} \parallel \langle 1,-1,0\rangle \times \langle 1,1,-1\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1,1,2\rangle.$$

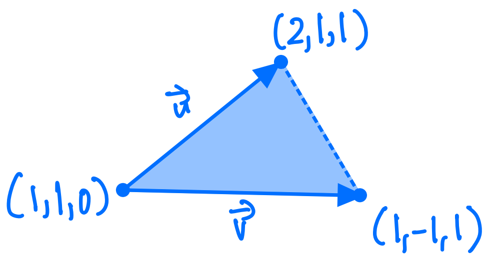
$\therefore$ Plane: $\langle 1,1,2\rangle \cdot \langle x,y,z\rangle = \langle 1,1,2\rangle \cdot \langle 0,0,1\rangle$

$= 0+0+2 = 2$

$$\boxed{x+y+2z = 2}.$$

13.4. Find the area of the triangle whose vertices are

$\langle 1,1,0\rangle, \langle 2,1,1\rangle, \langle 1,-1,1\rangle$.



$$\vec{u} = \langle 2,1,1\rangle - \langle 1,1,0\rangle = \langle 1,0,1\rangle$$

$$\vec{v} = \langle 1,-1,1\rangle - \langle 1,1,0\rangle = \langle 0,-2,1\rangle$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \langle 2,-1,-2\rangle$$

$$\text{Area} = \frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}\sqrt{4+1+4} = \boxed{\frac{3}{2}}$$

13.3. What is the relation between $\vec{u}$ and $\vec{v}$, if $\vec{u}\cdot\vec{v}=0$?

$\boxed{\vec{u}\perp\vec{v}}$.